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Deformations of spectral triples and their quantum isometry groups via monoidal equivalences

Liebrecht De Sadeleer

Supervisors:
Prof. dr. Johan Quaegebeur
Prof. dr. Pierre Bieliavsky

Dissertation presented in partial fulfillment of the requirements for the degree of Doctor of Science (PhD): Mathematics

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Examination committee:
Prof. dr. Marco Zambon, chair
Prof. dr. Johan Quaegebeur, supervisor
Prof. dr. Pierre Bieliavsky, supervisor
Prof. dr. Stefaan Vaes
Prof. dr. Karel Dekimpe
Prof. dr. Kenny De Commer
(Vrije Universiteit Brussel)
Prof. dr. Pierre Fima
(Université Denis-Diderot - Paris 7)

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Opgedragen aan mijn mama en al wie me de laatste vier jaren liefdevol omringde.

Dedicated to my mom and all those who lovingly surrounded me the last four years.

Dédié à ma mère et à tous ceux qui m'entouraient plein d'amour les quatre dernières années.

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Abstract

In this thesis, we develop a deformation procedure for spectral triples. The initial data for the deformation are:

- a spectral triple;
- a compact quantum group $\mathbb{G}$ acting algebraically and by orientation preserving isometries on the spectral triple;
- a unitary fiber functor on $\mathbb{G}$, or equivalently, a monoidal equivalence between $\mathbb{G}$ and a second compact quantum group.

This procedure is proven to be a generalization of the cocycle deformation which Goswami and Joardar introduced in [53]. Moreover, it is a proper generalization: we construct an example of our method which is not a deformation à la Goswami-Joardar. Finally, we prove that this deformation method is compatible with the notion of quantum isometry group: the quantum isometry group of a deformed spectral triple is a suitable deformation of the quantum isometry group of the initial quantum isometry group.

Beknopte samenvatting

In deze thesis ontwikkelen we een nieuwe deformatiemethode voor spectrale tripletten. De deformatie heeft als ingrediënten:

- een spectraal triplet;
- een compacte kwantumgroup $\mathbb{G}$ die algebraïsch, isometrisch en oriëntatie-bewarend werkt op het spectraal triplet;
- een unitaire vezel functor op $\mathbb{G}$, of equivalent, een monoïdale equivalentie tussen $\mathbb{G}$ en een tweede compacte kwantumgroep.

We bewijzen dat deze deformatiemethode een veralgemening is van de cocykeldeformatie die Goswami en Joardar voorstelden in [53]. Onze methode is echter een strikte veralgemening: we construeren in deze thesis een voorbeeld van een deformatie met onze methode die geen cocyckeldeformatie is.

Tot slot bewijzen we ook dat onze methode verenigbaar is met de kwantumisometriegroep: de kwantumisometriegroep van het gedeformeerde spectrale triplet blijkt een gepaste deformatie te zijn van de kwantumisometriegroep van het oorspronkelijke spectraal triplet.

Contents

Introduction

Introduction and motivation

Quantum spaces, quantum groups, quantum symmetries

In the 1980's, several simultaneous evolutions resulted in the development of quantum spaces and quantum groups. In physics, the standard model was being developed but it was a common opinion that, in order to be able to encompass a quantum version of gravity, one needed to push the consequences of the quantization of space-time into a new phase. Space itself needed to be quantized and different approaches were proposed, algebraic and analytical, but both following the same principles. The quantization of a finite resp. locally compact space consisted of replacing the commutative algebra of functions resp. C^*-algebra of continuous functions, on a space X to a non-commutative algebra resp. C^*-algebra, which were seen as the 'functions' resp. 'continuous functions' on the quantum space. The analytical approach is inspired by the Gelfand Naimark theorem. Different examples were constructed, algebraic as well as analytical, e.g. quantum planes by Manin [72] and the Podleś spheres [78].

In the theory of groups, there were also different attempts to find extensions of the notion of 'group'. For example, the Pontryagin duality of locally compact abelian groups was restricted to abelian groups and for an extension to all locally compact groups, one needed to extend the notion of 'group'. Already in the sixties, Kac [58] investigated "ring groups" and also later, in all points of view, this duality principle was important. Inspired by the quantum inverse scattering method Drinfel'd [46] and Jimbo [56] enlarged the set of Lie groups by defining a q-deformation of enveloping algebras of Lie algebras. For $q = 1$, the classical object reappeared. Also Faddeev, Reshetikhin and Takhtajan worked on this and expanded Drinfel'd's work in [47]. Manin [72, 73] suggested quantum groups arising as some kind of quantum symmetry object of quadratic algebras (i.e. quotients of free algebras by ideals generated by degree 2 homogeneous elements). This symmetry approach was later explored extensively in the context of analytical (topological) quantum groups.

Analytically, Woronowicz introduced the notion of compact matrix quantum group and generalized the Peter-Weyl representation theory and Tannaka Krein duality [101–104]. Also in this approach, different examples were constructed, e.g. the orthogonal and unitary universal quantum groups of van Daele and Wang [94]. Later Woronowicz extended his theory to general compact quantum groups [105] and in 2000 Vaes and Kustermans even generalized to locally compact quantum groups [65]. In the search for examples also here, a lot of research was done in finding quantum symmetry objects; we look into these examples later.

With the development of the concept of quantum groups, new questions arose in physics in the 1990's: Connes raised the question of quantum symmetries in 1995: if symmetries are important in classical mechanics and the concept of space is extended to quantum spaces, what is a good notion of quantum symmetry? Wang [95] was the first to give a partial answer for finite quantum spaces without extra (metric or differential) structure. Later, Banica, Bichon, Bhowmick and others investigated this notion further in different contexts [4, 6, 8, 9, 11, 22, 24, 25, 79]. An interesting survey is [7].

The progress in quantum spaces and their quantum symmetries extended to quantum spaces with extra structure: quantum metric spaces and quantum manifolds and there symmetries. Quantum metric spaces were studied in [31] and further developed in [69, 83–85] and different notions of quantum isometric action were formulated in [5] and [80]. Analogously to Wang's quantum automorphism groups for quantum spaces without metric or differential structure, the question arose whether there exists also a universal quantum symmetry group respecting this extra metric structure. The notion and existence of quantum isometry groups were also investigated (e.g. in [52]).

Alain Connes did a huge work in defining quantum manifolds (which are described by objects called spectral triples) resulting in the book [32]. The question of symmetries was then exported to non-commutative geometry: what are quantum isometries and quantum groups acting isometrically in this differential framework and is there a good notion of quantum isometry group of such a quantum manifold. Here Goswami and Bhowmick were pioneers and developed a notion of quantum isometry group in this context in [15, 50]. Further work and examples can be found in e.g. [10, 12–14, 16–21, 37, 51].

Monoidal equivalences: a tool...

In the study of quantum groups and their structure, the representation theory was always an interesting field of research as it gives a lot of information about the structure of quantum groups. Indeed, the Tannaka-Krein reconstruction theorem of Woronowicz [105] states that with the information of the representations as concrete monoidal category, one can reconstruct the whole quantum group. Therefore the structure of the representation category was and is a topic of great interest and research: e.g. Banica gave a complete description of the representation theory of the free analogues of $O(n)$ and $U(n)$ in [2] resp. [3].

In the algebraic description, representations are comodules and Ulbrich and Schauenburg described functors between categories of comodules. An important theorem of Ulbrich [90] (here stated in theorem 1.3.11) states that two Hopf algebras have equivalent strict monoidal categories of comodules if and only if there exists an algebra, called a bi-Galois object, realizing an algebraic link between the two Hopf algebras.

In the analytical description, this equivalence of representation categories has the name 'monoidal equivalence', developed in [27]. It gives a one-to-one correspondence between the representation spaces together with a one-to-one correspondence between their morphisms compatible with the monoidal structure of the representations.

It is striking how few explicit links there have been made in literature between the two approaches despite the equivalence of the two. One of the goals of the first two chapters is to make this link explicit, which we summarized in theorems 2.5.4 and 2.6.7.

It is that notion of monoidal equivalence that gives us the possibility to define a new procedure to construct deformations of spectral triples, which is the main goal of our thesis.

...to construct deformations

From the start of the development of the quantum notions, people wonder whether it is possible to obtain a deformation procedure transforming classical, known objects (spaces, groups, symmetry groups) to quantum objects.

Rieffel explored in [82] how to deform spaces with an isometric action of $\mathbb{R}^d$ to quantum spaces using oscillating integrals. Moreover, one can use this technique to deform spectral triples, as is proven in [15, 16]. Finally, it is a natural question whether there is a link between the quantum isometry group of the original spectral triple and the quantum isometry group of the deformed spectral triple. This turns out to be indeed the case: Wang constructed a method to deform quantum groups in this way in [96]. Moreover the quantum isometry group of the Rieffel-deformed spectral triple is the Wang-deformed quantum isometry group. Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with its quantum isometry group $\mathbb{G}$, one obtains a diagram as follows

$$\begin{tikzcd}
(\mathcal{A}, \mathcal{H}, D) \arrow[d] & \curvearrowleft & \mathbb{G} \arrow[d] \\
(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{D}) & \curvearrowleft & \widetilde{\mathbb{G}}
\end{tikzcd}$$

where $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{D})$ denotes the Rieffel-deformed spectral triple and $\widetilde{\mathbb{G}}$ the Wang-deformed quantum isometry group. This commuting diagram was inspiring for other deformation procedures. In [53] Goswami and Joardar extended the Rieffel deformation to deformations with arbitrary 2-cocycles.

It was already well known that a 2-cocycle on a Hopf algebra induces a deformation of that Hopf algebra and of the left and right comodule(-algebra)s. And also different steps were set in the direction of a deformation on the operator algebra level. Pioneers were Landstad [68] and Wasserman [99, 100] who studied ergodic actions of classical groups. Kasprzak generalised this to compact quantum groups and gave an alternative description of the Rieffel deformation of algebras in [60–62]. The equivalence of the two descriptions was proved by Neshveyev in [74]. It turned out that the Rieffel deformation for algebras was in fact n example of a cocycle deformation. Neshveyev, Tuset, Bhowmick and Sangha developed the cocylce deformation further in [20, 75].

Goswami and Joardar combined this knowledge knowledge with the Rieffel deformation of spectral triples to find a 2-cocycle deformation method for spectral triples. Moreover, this deformation procedure satisfied the commuting diagram property: the quantum isometry group of the deformed spectral is the deformed quantum isometry group.

The main goal of this thesis is to generalize the deformation procedure of Goswami and Joardar in [53]. In [43] we accomplish this goal and propose a deformation procedure using monoidal equivalences. It is indeed a generalization: in [27], the authors prove that from a dual 2-cocycle on a Hopf algebra H one can construct a monoidal equivalence between the compact quantum group associated to the original Hopf algebra and the compact quantum group associated to the deformed Hopf algebra. The deformation of spectral triples with the cocycle à la Goswami-Joardar turns out to be equal to the deformation of spectral triples with that constructed monoidal equivalence in our sense. However, the two methods are not completely equivalent: our method is a proper generalization of Goswami-Joardar's work; we give a concrete example based on the Podleś sphere, of a deformation not coming from a 2-cocycle. Finally, also this new deformation procedure deforms the quantum isometry group of the original spectral triple into the quantum isometry group of the deformed spectral triple.

Structure of the thesis

This thesis is structured as follows.

In the first chapter we describe the algebraic Hopf-Galois deformations of Hopf algebras, Hopf-*-algebras and their comodule(-algebra)s. We prove that for a Hopf algebra H_1 with a left Galois object B, we can construct a new Hopf algebra H_2 such that B is a $(H_1$-$H_2)$-bi-Galois object. This turns out to be equivalent with the equivalence of the respective categories of comodules. In the last section of that chapter, we prove that all the results are compatible with a possible *-structure. The first three sections of this chapter are not intended to be new work. The results are mostly taken from [87], [88], [86], [26], [23] and [39]. We tried to give a clear presentation of the results needed to make a link with the new work in chapters 3 to 6. The last section is partly new work. Not so much have been done in literature. We put work of [88], [86], [26], [23] and [39] together and extended where needed.

In the second chapter, we give an alternative description following an analytical way. We first recall some basic notions and results about compact and discrete quantum groups and actions of compact quantum groups. Then we recall the work of [27] on monoidal equivalences and we prove that the algebraic theory of the first section is equivalent to the analytical theory of monoidal equivalences. We do this in theorems 2.5.4 and 2.6.7. It is this analytical approach that we will use to define a deformation of spectral triples in the third chapter.

The third chapter contains the main result of this thesis. We first look into spectral triples as basis of non-commutative geometry and compact quantum groups acting on them. In a second section, we introduce a "box product" for Hilbert spaces, a tool we need to define the deformed spectral triple. The heart of the chapter consists in stating and proving our main theorem (theorem 3.3.8):

Main Result A. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a compact spectral triple and let $\mathbb{G}_1 = (C(\mathbb{G}_1), \Delta_1)$ be a compact quantum group acting algebraically and by orientation-preserving isometries on $(\mathcal{A}, \mathcal{H}, D)$ with a unitary representation U. Moreover let $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$ be a monoidal equivalence between $\mathbb{G}_1$ and a compact quantum group $\mathbb{G}_2$.*

Then there exist a spectral triple $(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$ such that $\mathbb{G}_2$ acts algebraically and by orientation-preserving isometries on the new spectral triple.
$(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$ is called the monoidal deformation of $(\mathcal{A}, \mathcal{H}, D)$ and $\mathbb{G}_2$ the deformation of $\mathbb{G}_1$.

Finally, we prove that our deformation is 'reversible': if we apply the same construction with as data, the deformed spectral triple with deformed quantum group and the inverse monoidal equivalence, we get back our initial spectral triple and compact quantum group acting on it.

In the fourth chapter, we prove that the 2-cocycle deformation of spectral triples of Goswami and Joardar fits into the framework of the new method developed in chapter three as an example. We do that by having a closer look at unitary fiber functors and monoidal equivalences with the extra property that the dimensions of the irreducible representations are preserved under the monoidal equivalence. Unitary fiber functors which satisfy this condition will be called dimension-preserving. A monoidal deformation arising from a dimension-preserving unitary fiber functor will be called a dimension-preserving monoidal deformation. Bichon et al. proved in [27] that dimension-preserving unitary fiber functors are in one-to-one correspondence with 2-cocycles on the dual quantum group. Using this, we will prove that dimension-preserving monoidal deformation is equivalent to the cocycle deformation introduced in [53]. We have the following theorem (theorem 4.4.1):

Main Result B. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple, $\mathbb{G}$ a compact quantum group acting on it algebraically and by orientation-preserving isometries and let φ be a dimension-preserving monoidal equivalence between $\mathbb{G}$ and a compact* *quantum group $\mathbb{G}'$. Then there exists a 2-cocycle σ such that the Goswami-Joardar deformation $(\mathcal{A}_{\sigma^{-1}}\#\mathbb{C}, \mathcal{H}, D)$ and the monoidal deformation $(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$ are isomorphic as spectral triples.*

The goal of the fifth chapter is to prove that our deformation procedure strictly generalizes that of Goswami and Joardar. In order to do so, we construct an example of a monoidal deformation coming from a non-dimension-preserving monoidal equivalence. We will use the Podleś spheres introduced in [78] and the spectral triple on it defined in [36]. The compact quantum group acting on it isometrically will be $SU_q(2)$ the quantized version of $SU(2)$ defined in [104]. The study of [27] about the monoidal equivalences of $SU_q(2)$ tells us which monoidal equivalences are non-dimension-preserving. The main theorem of this chapter is theorem 5.2.1:

Main Result C. *The monoidal deformation with initial data:*

- *the spectral triple on the Podleś sphere,*
- *a non-dimension-preserving monoidal equivalence $\varphi : SU_q(2) \to \mathbb{G}$ between quantized $SU(2)$ and a suitable compact quantum group $\mathbb{G}$ and*
- *the algebraic action of $SU_q(2)$ on the spectral triple on the Podleś sphere by orientation preserving isometries*

is not a 2-cocycle deformation à la Goswami-Joardar [53].

Finally in the sixth and last chapter, we focus on quantum isometry groups. In the first section we introduce some basics about quantum isometry groups, e.g. a volume form coming form an operator R. After that, we develop some tools in the second section. If $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$ is a monoidal equivalence, we construct a procedure to induce monoidal equivalence between certain Woronowicz C^*-subalgebras and between certain quantum supergroups $\mathbb{H}_1 \to \mathbb{H}_2$ (i.e. if $\mathbb{G}_1$ resp. $\mathbb{G}_2$ is a quantum subgroup of $\mathbb{H}_1$ resp. $\mathbb{H}_2$).
In section 6.3 we prove that the deformation of the quantum isometry group of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$, is the quantum isometry group of $(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$. We have the important theorem 6.3.4.

Main Result D. *The quantum isometry group of the monoidal deformation of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is the deformation of the quantum isometry group of $(\mathcal{A}, \mathcal{H}, D)$.*

Again we get a commutative diagram:

$$\begin{tikzcd}
(\mathcal{A}, \mathcal{H}, D) \arrow[d] \arrow[r, phantom, "\curvearrowleft"] & \mathrm{QISO}_R(\mathcal{A}, \mathcal{H}, D) \arrow[d] \\
(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{D}) \arrow[r, phantom, "\curvearrowleft"] & \mathrm{QISO}_{\tilde{R}}(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})
\end{tikzcd}$$

where the quantum isometry group is denoted by $\mathrm{QISO}_R(\mathcal{A}, \mathcal{H}, D)$.

We end this thesis by stating a conclusion and some remarks for further research.

Concerning credits

As already mentioned, in the first chapter, there is few new work. The first three sections are a concise review of work of [87], [88], [86], [26], [23] and [39]. The last section is somehow new: some small gaps are filled, but inspiration and results are taken from the above references.

In the second chapter we recall the theory of [102], [105]. Also [70] is used. Some proofs are elaborated or new, but inspiration was always taken from these (or if indicated, other) references. Also [27] is frequently used here, however some left-right changes are made and proofs are adapted to these changes. Whenever we do this, it is indicated. Also the proof of theorem 2.5.4 is partly new. The theorem is known (as one can find in introductions of several papers) but was, to my knowledge, never made explicit.

The third chapter is new. After recalling some basic non-commutative geometry notions from [32] and [15], we develop, step by step the deformation procedure.

In the fourth chapter we rely on [27] for the equivalence between the (discrete quantum group) 2-cocycles and the dimension-preserving monoidal equivalences. Also the algebraic theory of dual 2-cocycles is well known, for example [64, 71]. The link between a dimension-preserving deformation and a 2-cocycle deformation à la Goswami-Joardar however is new work.

The fifth chapter starts with recalling the work [27] on monoidal equivalences of $SU_q(2)$. Also the work of Podleś on the Podleś sphere [79] and the spectral triple on it [36] is known. The application of the method developed in the third chapter on this data is however new work.

In the last chapter, we introduce the theory of quantum isometry groups of [15] and use Wang's work on quantum quotient groups. Apart from that, everything is new work.

Chapter 1

Algebraic Hopf-Galois deformations

In this chapter we describe the algebraic Hopf-Galois deformations of Hopf algebras, Hopf *-algebras and their comodule(-algebra)s. The chapter is structured as follows. In the first section we recall some preliminaries on Hopf algebras. The second section is the heart of this chapter. We describe the algebraic deformation by constructing a new Hopf algebra via the Galois object. It will be clear that this method is only interesting in the non-classical case: if the Hopf algebra and Galois object are both commutative, the new Hopf algebra is isomorphic to the original one. In the third section, we have a closer look at the extra structure of the set of Hopf algebras and of Galois objects induced by this deformation procedure. The categorical description gives insight here. We will also see that two Hopf algebras are Hopf-Galois deformations of each other if and only if their respective strict monoidal categories of comodules are equivalent. We will see that this equivalence is also crucial in the second chapter. Finally in the last section we apply all of this to Hopf *-algebras in order to prepare an other point of view for the next chapter.

This chapter is not intended to be new work. The results are mostly taken from [87], [88], [86], [26], [23], [39] [38], sometimes we give an alternative presentation. The work in section 1.4 is partly new and we give a new way of presentation. For the whole chapter, we tried to give a clear presentation of the results needed to make a link with the new work in chapters 3 to 6.

1.1 Preliminaries on Hopf algebras and Hopf comodules

Before we start this chapter, we will recall some terminology and notation. For a Hopf algebra H, the coproduct, counit and antipode will be denoted by Δ, ε and S resp. We also use the Sweedler notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ where the summation is implied. A left, resp. right H-comodule is a vector space A endowed with a linear map $\alpha : A \to H \odot A$ resp. $\alpha : A \to A \odot H$ satisfying $(\Delta \otimes \mathrm{id})\alpha = (\mathrm{id} \otimes \alpha)\alpha$ resp. $(\alpha \otimes \mathrm{id})\alpha = (\mathrm{id} \otimes \Delta)\alpha$. The map α is then called a coaction of H on V. Also here, we use a Sweedler notation: $\alpha(a) = a_{(-1)} \odot a_{(0)}$ for a left coaction and $\alpha(a) = a_{(0)} \odot a_{(1)}$ for a right coaction. If A is an algebra and α is multiplicative, A is called an H-comodule-algebra. If A and B are a right resp. left H-comodule-algebra with resp. coactions α and β, $A \underset{H}{\boxdot} B$ will denote the algebra $\{z \in A \odot B | (\alpha \otimes \mathrm{id})(z) = (\mathrm{id} \otimes \beta)(z)\}$. The algebraic tensor product is denoted by $\odot$. For a left resp. right H -comodule (algebra) B, the tensor product $B \odot B$ has a natural H-comodule structure via the map

$$\lambda : B \odot B \to H \odot B \odot B : b \otimes b' \mapsto b_{(-1)} b'_{(-1)} \otimes b_{(0)} \otimes b'_{(0)}$$

resp. $\lambda : B \odot B \to B \odot B \odot H : b \otimes b' \mapsto b_{(0)} \otimes b'_{(0)} \otimes b_{(1)} b'_{(1)}$. We call the coaction the codiagonal coaction of H on $B \odot B$. Elements $z \in B \odot B$ satisfying $\lambda(z) = 1_H \odot z$ (resp. $\lambda(z) = z \otimes 1_H$) are called the coinvariants and we denote by ${}^{coH}(B \odot B)$ resp. $(B \odot B)^{coH}$ the set of coinvariants.

If H is a Hopf *-algebra, an H-*-comodule is a comodule, which has $\alpha(a^*) = \alpha(a)^*$. Moreover, for an algebra B, we will use the notation $\mathrm{id}_{op} : B \to B^{op} : b \to b^{op}$ and also $B^{op} \to B : b^{op} \to b$. Furthermore for a, b elements of an algebra B, we use the map $\sigma : B \odot B \to B \odot B : a \otimes b \mapsto b \otimes a$.
Finally, we will use a summation convention: if a sub- or superscript appears twice or more, the summation is implied. For example $b_i \otimes b'_i$ is used for $\sum_i b_i \otimes b'_i$.

For more basic material on Hopf algebras, we refer to [63, 64, 71].

1.2 Hopf-Galois deformation for Hopf algebras

The goal of this second section is to develop the algebraic deformation procedure for Hopf algebras. We define first the concept of Galois object, investigate its classical meaning and look at some calculation results. In the second subsection, we construct the deformed Hopf algebra with a Galois object. The theorems stated in this section are adapted from [87], where the author works mostly with right Galois objects. As it is more convenient for us to work with left Galois objects (in order to make the link with the other chapters), this adaptation had to made. Nevertheless, we give credits to [87].

1.2.1 Galois objects on Hopf algebras

Classical example. *For some definitions and theorems in this chapter, we will make the link with the classical picture where $H = C(G)$ and $B = C(X)$ for a finite group G and a finite space X. G acts as a group on X.*

Definition 1.2.1 ([87]). *Let H be a Hopf algebra and B a unital H-comodule-algebra with $\beta : B \to H \otimes B$ the left coaction of H on B. We call B a left H-Galois object if the linear map*

$$T_\beta : B \odot B \to H \odot B : b \otimes b' \mapsto \beta(b)(1_H \otimes b'),$$

called a Galois map, is a bijection. Analogously, we call B with $\tilde{\beta} : B \to B \odot H$ a right H-Galois object if the linear map

$$R_{\tilde{\beta}} : B \odot B \to B \odot H : b \otimes b' \mapsto (b \otimes 1_H)\tilde{\beta}(b')$$

is a bijection. For further use, we use the notation

$$\gamma : H \to B \odot B : h \mapsto T_\beta^{-1}(h \otimes 1_B) = \sum_i l_i(h) \otimes r_i(h).$$

For notational convenience, we will sometimes choose to not write the summation explicitly. Note that if B is commutative, then T_β is not only linear, but also multiplicative.

Classical example. *In the classical case, for an action $\beta : G \times X \to X : (g, x) \mapsto \beta_g(x) = g \cdot x$ we construct the map*

$$T^c_\beta : G \times X \to X \times X : (g, x) \mapsto (g \cdot x, x).$$

If T^c_β is bijective, we call (G, X) a left Galois pair. (Analogously, we can define (X,G) to be a right Galois pair.) The map γ can be translated to a map

$$\gamma^c : X \times X \to G : (y, x) \mapsto g$$

where g is such that $y = g \cdot x$.

There exists also a non-classical easy example. If H is a non-commutative Hopf algebra, (H, Δ) is a left and also a right H-Galois object. It is easy to see that

$$T_\Delta : H \odot H \to H \odot H : h \otimes h' \to h_{(1)} \otimes h_{(2)} h'$$

and

$$R_\Delta : H \odot H \to H \odot H : h \otimes h' \to h h'_{(1)} \otimes h'_{(2)}$$

have respective inverses

$$T'_\Delta : H \odot H \to H \odot H : h \otimes h' \to h_{(1)} \otimes S(h_{(2)}) h'$$

and

$$R'_\Delta : H \odot H \to H \odot H : h \otimes h' \to h S(h'_{(1)}) \otimes h'_{(2)}$$

proving that indeed H is a left and right H-Galois object.

Moreover, it turns out that the coaction on a Galois object is automatically ergodic.

Lemma 1.2.2. *Let H be a Hopf algebra and (B, β) a left H-Galois object. Then β is ergodic i.e. $\{b \in B | \beta(b) = 1 \otimes b\} = \mathbb{C}1_B$.*

Proof. Suppose b is an element of B such that $\beta(b) = 1 \otimes b$ and $b' \in B$. Then $T_\beta(b \otimes b') = \beta(b)(1 \otimes b') = 1 \otimes bb' = T_\beta(1 \otimes bb')$ and hence T_β can't be bijective unless $b \in \mathbb{C}1_B$. $\square$

For further use, we make some calculations. In [87] these calculations are done for right Galois objects, we do it for left Galois objects.

Lemma 1.2.3 ([87]). *Let H be a Hopf algebra, B a left H-Galois object and $\gamma : H \to B \odot B$ as in definition 1.2.1. Then for $g, h \in H$ and $b \in B$, one has*

1. $m \circ \gamma(h) = \varepsilon(h) 1_B$,
2. $(\mathrm{id} \otimes \gamma)\Delta(h) = (\beta \otimes \mathrm{id})\gamma(h)$,
3. $(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \beta)\gamma(h) = (S \otimes \gamma)\sigma \circ \Delta(h)$,
4. $\gamma(gh) = (m \otimes m\sigma)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\gamma(g) \otimes \gamma(h))$,
5. $b \otimes 1_B = (\mathrm{id} \otimes m)(\gamma \otimes \mathrm{id}_B)\beta(b)$,
6. $\gamma(1_H) = 1_B \otimes 1_B$,

7. $T_\beta^{-1}(h \otimes b) = \gamma(h)(1 \otimes b)$.

Remark 1.2.4. *The above results can be written using the Sweedler notation and the notation* $\gamma(h) = l_i(h) \otimes r_i(h)$*:*

1. $l_i(h) r_i(h) = \varepsilon(h) 1_B$,
2. $h_{(1)} \otimes l_i(h_{(2)}) \otimes r_i(h_{(2)}) = l_i(h)_{(-1)} \otimes l_i(h)_{(0)} \otimes r_i(h)$,
3. $r_i(h)_{(-1)} \otimes l_i(h) \otimes r_i(h)_{(0)} = S(h_{(2)}) \otimes l_i(h_{(1)}) \otimes r_i(h_{(1)})$,
4. $\gamma(gh) = l_i(g) l_j(h) \otimes r_j(h) r_i(g)$,
5. $b \otimes 1_B = l_i(b_{(-1)}) \otimes r_i(b_{(-1)}) b_{(0)}$,
6. $l_i(1_H) \otimes r_i(1_H) = 1_B \otimes 1_B$,
7. $T_\beta^{-1}(h \otimes b) = l_i(h) \otimes r_i(h) b$.

Proof of Lemma 1.2.3.

1. We have $h \otimes 1_B = T_\beta(\gamma(h)) = \beta(l_i(h))(1 \otimes r_i(h))$ and hence $\varepsilon(h) 1_B = (\varepsilon \otimes \mathrm{id})\beta(l_i(h)) r_i(h) = l_i(h) r_i(h) = m \circ \gamma(h)$.

2. As $(T_\beta \circ \gamma)(h) = h \otimes 1_B$, it suffices to prove that $(\mathrm{id} \otimes T_\beta)(\beta \otimes \mathrm{id})\gamma(h) = \Delta(h) \otimes 1_B$. We have:

$$\begin{aligned}
(\mathrm{id} \otimes T_\beta)(\beta \otimes \mathrm{id})\gamma(h) &= (\mathrm{id} \otimes \mathrm{id} \otimes m)(\mathrm{id} \otimes \beta \otimes \mathrm{id})(\beta \otimes \mathrm{id})(\gamma(h)) \\
&= (\mathrm{id} \otimes \mathrm{id} \otimes m)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\beta \otimes \mathrm{id}) \circ \gamma(h) \\
&= (\Delta \otimes \mathrm{id})(\mathrm{id} \otimes m)(\beta \otimes \mathrm{id}) \circ \gamma(h) \\
&= (\Delta \otimes \mathrm{id}) \circ T_\beta \circ \gamma(h) \\
&= (\Delta \otimes \mathrm{id})(h \otimes 1_B) \\
&= \Delta(h) \otimes 1_B.
\end{aligned}$$

3. Applying the map $(\mathrm{id}_H \otimes T_\beta)$ on both sides, the equality is equivalent to

$$(\mathrm{id}_H \otimes T_\beta)(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \beta)\gamma(h) = (S \otimes \mathrm{id})\sigma\Delta(h) \otimes 1$$

or,

$$(\mathrm{id}_H \otimes T_\beta)\Big(r_i(h)_{(-1)} \otimes l_i(h) \otimes r_i(h)_{(0)}\Big) = S(h_{(2)}) \otimes h_{(1)} \otimes 1.$$

Introducing the map:

$$\alpha : H \otimes B \to H \otimes H \otimes B : (h \otimes b) \mapsto S(h_{(2)})b_{(-1)} \otimes h_{(1)} \otimes b_{(0)},$$

we get $S(h_{(2)}) \otimes h_{(1)} \otimes 1 = \alpha(h \otimes 1) = \alpha(T_\beta(\gamma(h)))$. On the other hand, we have

$$\begin{aligned} \alpha(T_\beta(x \otimes y)) &= \alpha\Big(x_{(-1)} \otimes x_{(0)}y\Big) \\ &= S(x_{(-2)})x_{(-1)}y_{(-1)} \otimes x_{(-3)} \otimes x_{(0)}y_{(0)} \\ &= y_{(-1)} \otimes \varepsilon(x_{(-1)})x_{(-2)} \otimes x_{(0)}y_{(0)} \\ &= y_{(-1)} \otimes x_{(-1)} \otimes x_{(0)}y_{(0)} \\ &= (\mathrm{id}_H \otimes T_\beta)\Big(y_{(-1)} \otimes x \otimes y_{(0)}\Big) \end{aligned}$$

for every $x, y \in B$, and by linearity,

$$\alpha\Big(T_\beta\big(l_i(h) \otimes r_i(h)\big)\Big) = (\mathrm{id}_H \otimes T_\beta)\Big(r_i(h)_{(-1)} \otimes l_i(h) \otimes r_i(h)_{(0)}\Big).$$

Putting everything together, we obtain the desired equality. Indeed:

$$\begin{aligned} (\mathrm{id}_H \otimes T_\beta)\Big(r_i(h)_{(-1)} \otimes l_i(h) \otimes r_i(h)_{(0)}\Big) &= \alpha\big(T_\beta(\gamma(h))\big) \\ &= S(h_{(2)}) \otimes h_{(1)} \otimes 1 \end{aligned}$$

which proves the statement.

4. It suffices to proof that

$$gh \otimes 1 = T_\beta\Big(l_i(g)l_j(h) \otimes r_j(h)r_i(g)\Big).$$

We have,

$$\begin{aligned} & T_\beta\Big(l_i(g)l_j(h) \otimes r_j(h)r_i(g)\Big) \\ = \ & l_i(g)_{(-1)}l_j(h)_{(-1)} \otimes l_i(g)_{(0)}l_j(h)_{(0)}r_j(h)r_i(g) \\ = \ & \Big(l_i(g)_{(-1)} \otimes l_i(g)_{(0)}\Big)\Big(l_j(h)_{(-1)} \otimes l_j(h)_{(0)}r_j(h)\Big)\Big(1 \otimes r_i(g)\Big) \\ = \ & \Big(l_i(g)_{(-1)} \otimes l_i(g)_{(0)}r_i(g)\Big)(h \otimes 1) \\ = \ & gh \otimes 1 \end{aligned}$$

where we used $T_\beta(\gamma(h)) = h \otimes 1_B$. This proves that indeed $\gamma(gh) = l_i(g)l_j(h) \otimes r_j(h)r_i(g)$.

5. Taking T_β of the left side, we get:

$$\begin{aligned}
&T_\beta(\mathrm{id}_B \otimes m)(\gamma \otimes \mathrm{id}_B)\beta(b)\\
&\quad = (\mathrm{id}_H \otimes m)(\beta \otimes \mathrm{id}_B)(\mathrm{id}_B \otimes m)(\gamma \otimes \mathrm{id}_B)\beta(b)\\
&\quad = (\mathrm{id} \otimes m)(\mathrm{id} \otimes \mathrm{id} \otimes m)((\beta \otimes \mathrm{id})\gamma \otimes \mathrm{id}_B)\beta(b)\\
&\quad = (\mathrm{id} \otimes m)(\mathrm{id} \otimes m \otimes \mathrm{id})((\mathrm{id} \otimes \gamma)\Delta \otimes \mathrm{id}_B)\beta(b)\\
&\quad = (\mathrm{id} \otimes m)((\mathrm{id} \otimes m\gamma)\Delta \otimes \mathrm{id}_B)\beta(b)\\
&\quad = \beta(b)\\
&\quad = T_\beta(b \otimes 1)
\end{aligned}$$

and as T_β is bijective, this proves that $b \otimes 1_B = (\mathrm{id} \otimes m)(\gamma \otimes \mathrm{id}_B)\beta(b)$.

6. Taking $b = 1$ in the previous calculations, we get

$$1_B \otimes 1_B = l_i(1_B) \otimes r_i(1_B)1_B = \gamma(1_B).$$

7. Let $h \in H$ and $b \in B$ arbitrary. Denote $L_\beta(h \otimes b) = \gamma(h)(1 \otimes b)$. Then, we have, using the above calculation rules:

$$\begin{aligned}
T_\beta \circ L_\beta(h \otimes b) &= T_\beta\big(l_i(h) \otimes r_i(h)b\big)\\
&= l_i(h)_{(-1)} \otimes l_i(h)_{(0)} r_i(h)b\\
&= h_{(1)} \otimes l_i(h_{(2)})r_i(h_{(2)})b\\
&= h_{(1)}\varepsilon(h_{(2)}) \otimes b\\
&= h \otimes b
\end{aligned}$$

and

$$\begin{aligned}
L_\beta \circ T_\beta(b \otimes b') &= \gamma(b_{(-1)})(1 \otimes b_{(0)}b')\\
&= l_i(b_{(-1)}) \otimes r_i(b_{(-1)})b_{(0)}b'\\
&= b \otimes b'
\end{aligned}$$

and we can conclude that $L_\beta = T_\beta^{-1}$.

□

Remark 1.2.5. *By (4) and (6) of Lemma 1.2.3, $\gamma : H \to B \odot B^{op}$ is a unital algebra morphism.*

1.2.2 Bi-Galois objects and Hopf-Galois deformation for Hopf algebras

In this second subsection, we prove that a Galois object, initially giving information about only one Hopf algebra, in fact makes a link between two Hopf algebras.

Definition 1.2.6 ([87]). *Let $H, \tilde{H}$ be Hopf algebras, B a unital algebra with β a left coaction of H on B and $\tilde{\beta}$ a right coaction of $\tilde{H}$ on B. We call $(B, \beta, \tilde{\beta})$ a (H-$\tilde{H}$)-bi-Galois object if (B, β) is a left H-Galois object and $(B, \tilde{\beta})$ is a right $\tilde{H}$-Galois object and β and $\tilde{\beta}$ commute i.e.*

$$(\mathrm{id}_H \otimes \tilde{\beta})\beta = (\beta \otimes \mathrm{id}_{\tilde{H}})\tilde{\beta}.$$

Classical example. *Classically, this means the following: we have two groups G_1 and G_2 such that (G_1, X) is a left and (X, G_2) a right Galois pair and such that the actions commute. We call (G_1, X, G_2) a bi-Galois triple.*

Proposition 1.2.7. *Let G_1 and G_2 be two groups such that (G_1, X, G_2) is a bi-Galois triple. Then there exists a group isomorphism*

$$\varphi : G_1 \to G_2.$$

Proof. Let $\beta_1 : G_1 \times X \to X$ and $\beta_2 : X \times G_2 \to X$ be the two respective actions. Then, we have a map:

$$\varphi' : G_1 \times X \to X \times G_2 : (g_1, x) \mapsto (T^c_{\beta_2})^{-1}(T^c_{\beta_1}(g_1^{-1}, x)) = (g_1^{-1} \cdot x, g_2)$$

where g_2 is such that $g_1 \cdot x = x \cdot g_2$. This is indeed bijective as composition of two bijections. Now choose an arbitrary $x_0 \in X$ and let

$$\varphi : G_1 \to G_2 : g_1 \mapsto \pi_2 \varphi'(g_1, x_0)$$

where π_2 is the projection onto the second coordinate. Note now that $\varphi'^{-1}(x_0, g_2) = (g_1, x_0 \cdot g_2)$ where $g_1 \in G_1$ is such that $g_1 \cdot x_0 = x_0 \cdot g_2$. Therefore the map $G_2 \to G_1 : g_2 \to \pi_1 \varphi'^{-1}(x_0, g_2)$ is the inverse of φ and hence, it is a bijection. To prove that it is a group morphism, note that $\varphi(g)$ is the unique element in G_2 such that $g \cdot x_0 = x_0 \cdot \varphi(g)$. Hence we have

$$x_0 \cdot \varphi(gg') = gg' \cdot x_0 = g \cdot x_0 \cdot \varphi(g') = x_0 \cdot \varphi(g)\varphi(g')$$

where we used that both actions commute and by unicity, we can conclude that $\varphi(gg') = \varphi(g)\varphi(g')$ for every $g, g' \in G_1$ and hence φ is a group isomorphism. $\square$

It is good to note that the map φ depends on the choice of x_0 and in this sense, it is not a canonical isomorphism. Moreover, this proposition shows that in the classical case (i.e. where H and B are both commutative), the concept of bi-Galois object is not interesting. It amounts always to an isomorphism. If B or both B and H are not commutative, interesting things do show up.

The following theorem is the main result of this section. It states that for a given Hopf algebra and a given (left or right) Galois object for it, there exists a new Hopf algebra such that the Galois object is a bi-Galois object for both. In the next section, we will prove that it is indeed a deformation: using the deformed data, you can deform again and get back to the original data.

Theorem 1.2.8 ([87]). *Let H be a Hopf algebra and (B, β) a left H-Galois object. Then there exist a (unique) Hopf algebra $(\widetilde{^BH}, \tilde{\Delta})$ which makes B a $((H, \beta)$-$(\widetilde{^BH}, \tilde{\beta}))$-bi-Galois object. We will call $\widetilde{^BH}$ a Hopf-Galois deformation of H along B.*

As the proof of the theorem consists of a concrete construction of the deformed Hopf algebra, it is quiet involved. We first state and prove some lemma's: after the definition of $\widetilde{^BH}$, the first lemma proves it is an algebra. The second introduces the morphism $\tilde{\beta}$, the third proves that $\widetilde{^BH}$ is a Hopf algebra and the last proves that $\tilde{\beta}$ is a right coaction of $\widetilde{^BH}$ on B. Finally, in the proof of the theorem, we will prove $R_{\tilde{\beta}}$ is bijective and the two actions commute.

Definition 1.2.9. *Let H be a Hopf algebra and (B, β) a left H-Galois object. We denote by $\widetilde{^BH}$ the set of coinvariants of the codiagonal coaction of H (i.e. $\widetilde{^BH} = \{b_i \otimes b'_i \in B \odot B \mid b_{i(-1)}b'_{i(-1)} \otimes b_{i(0)} \otimes b'_{i(0)} = 1 \otimes b_i \otimes b'_i\}$).*

We will use the notation $\lambda : B \odot B \to H \odot B \odot B : b_i \otimes b'_i \mapsto b_{i(-1)}b'_{i(-1)} \otimes b_{i(0)} \otimes b'_{i(0)}$ for the codiagonal coaction of H on $B \odot B$. Note that this is a coaction on the vector space $B \odot B$, but not on the algebra $B \odot B$.

Lemma 1.2.10. *$\widetilde{^BH}$ is a subalgebra of $B^{op} \odot B$.*

Proof. For $b_i \otimes b'_i, c_j \otimes c'_j \in \widetilde{{}^B H}$ (with implicit summation) we get:

$$\begin{aligned}
&\lambda(c_j b_i \otimes b'_i c'_j) \\
&= c_{j(-1)} b_{i(-1)} b'_{i(-1)} c'_{j(-1)} \otimes c_{j(0)} b_{i(0)} \otimes b'_{i(0)} c'_{j(0)} \\
&= \big(c_{j(-1)} \otimes c_{j(0)} \otimes 1\big)\big(b_{i(-1)} b'_{i(-1)} \otimes b_{i(0)} \otimes b'_{i(0)}\big)\big(c'_{j(-1)} \otimes 1 \otimes c'_{j(0)}\big) \\
&= c_{j(-1)} c'_{j(-1)} \otimes c_{j(0)} b_i \otimes b'_i c'_{j(0)} \\
&= (1 \otimes 1 \otimes b'_i)(c_{j(-1)} c'_{j(-1)} \otimes c_{j(0)} \otimes c'_{j(0)})(1 \otimes b_i \otimes 1) \\
&= 1 \otimes c_j b_i \otimes b'_i c'_j
\end{aligned}$$

and hence indeed, $\widetilde{{}^B H}$ is a subalgebra of $(B^{op} \odot B)$. □

Lemma 1.2.11. *Let H, (B, β) and $\widetilde{{}^B H}$ be as above. Then there exists a unital morphism $\tilde{\beta} : B \to B \odot \widetilde{{}^B H}$, namely $\tilde{\beta} = (\gamma \otimes \mathrm{id}_B)\beta$.*

Proof. We define: $\tilde{\beta}_0 : B \to B \odot B^{op} \odot B : b \mapsto (\gamma \otimes \mathrm{id}_B)\beta(b) = l_i(b_{(-1)}) \otimes r_i(b_{(-1)}) \otimes b_{(0)}$. Note first that as $\beta : B \to H \odot B$ and $\gamma : H \to B \odot B^{op}$ are unital algebra morphisms, $\tilde{\beta}_0$ is a unital algebra morphism $B \to B \odot B^{op} \odot B$. Hence it suffices to prove that $\tilde{\beta}_0(b) \in B \odot \widetilde{{}^B H}$, which means that

$$l_i(b_{(-2)}) \otimes r_i(b_{(-2)})_{(-1)} b_{(-1)} \otimes r_i(b_{(-2)})_{(0)} \otimes b_{(0)} = l_i(b_{(-1)}) \otimes 1_H \otimes r_i(b_{(-1)}) \otimes b_{(0)}.$$

Indeed, we have:

$$\begin{aligned}
&l_i(b_{(-2)}) \otimes r_i(b_{(-2)})_{(-1)} b_{(-1)} \otimes r_i(b_{(-2)})_{(0)} \otimes b_{(0)} \\
&= l_i(b_{(-3)}) \otimes S(b_{(-2)}) b_{(-1)} \otimes r_i(b_{(-3)}) \otimes b_{(0)} \\
&= l_i(b_{(-2)}) \otimes \varepsilon(b_{(-1)}) 1_H \otimes r_i(b_{(-2)}) \otimes b_{(0)} \\
&= l_i(b_{(-1)}) \otimes 1_H \otimes r_i(b_{(-1)}) \otimes b_{(0)}
\end{aligned}$$

and hence, $\tilde{\beta}_0(B) \subset B \odot \widetilde{{}^B H}$. We can define $\tilde{\beta} : B \to B \odot \tilde{H} : b \mapsto \tilde{\beta}_0(b)$. This concludes the proof.

□

Lemma 1.2.12. *Let H, (B, β) and $\widetilde{{}^B H}$ be as above. Then*

1. $\tilde{\Delta} = \mathrm{id}_{B^{op}} \otimes \tilde{\beta}$ *is a coproduct for* $\widetilde{{}^B H}$,
2. $\tilde{\varepsilon} = m_{B \odot B}$ *is a counit for* $\widetilde{{}^B H}$ *and*
3. $\tilde{S}(b_i \otimes b'_i) = l_j(b_{i(-1)}) b'_i r_j(b_{i(-1)}) \otimes b_{i(0)}$ *is an antipode for* $\widetilde{{}^B H}$

making $(\widetilde{{}^B H}, \tilde{\Delta}, \tilde{\varepsilon}, \tilde{S})$ *a Hopf algebra.*

Proof. 1. Define $\Delta_0 : \widetilde{{}^B H} \to B^{op} \odot B \odot \widetilde{{}^B H} : b_i \otimes b'_i \mapsto b_i \otimes \tilde{\beta}(b'_i)$ which is well defined as an algebra morphism by lemma 1.2.10 and by lemma 1.2.11. First we will prove that $\Delta_0(\widetilde{{}^B H}) \subset \widetilde{{}^B H} \odot \widetilde{{}^B H}$. We have, using $b_{i(-1)} b'_{i(-2)} \otimes b_{i(0)} \otimes b'_{i(-1)} \otimes b'_{i(0)} = 1_H \otimes b_i \otimes b'_{i(-1)} \otimes b'_{i(0)}$,

$$
\begin{aligned}
&(\lambda \otimes \mathrm{id}_{\widetilde{{}^B H}})(\Delta_0(b_i \otimes b'_i)) \\
&= \lambda(b_i \otimes l_i(b'_{i(-1)})) \otimes r_i(b'_{i(-1)}) \otimes b'_{i(0)} \\
&= b_{i(-1)} l_i(b'_{i(-1)})_{(-1)} \otimes b_{i(0)} \otimes l_i(b'_{i(-1)})_{(0)} \otimes r_i(b'_{i(-1)}) \otimes b'_{i(0)} \\
&= b_{i(-1)} b'_{i(-2)} \otimes b_{i(0)} \otimes l_i(b'_{i(-1)}) \otimes r_i(b'_{i(-1)}) \otimes b'_{i(0)} \\
&= 1_H \otimes b_i \otimes l_i(b'_{i(-1)}) \otimes r_i(b'_{i(-1)}) \otimes b'_{i(0)} \\
&= 1_H \otimes \Delta_0(b_i \otimes b'_i),
\end{aligned}
$$

so that we can conclude that indeed $\Delta_0(\widetilde{{}^B H}) \subset \widetilde{{}^B H} \odot \widetilde{{}^B H}$ and the map $\tilde{\Delta}$ is well defined.

Furthermore, note that:

$$
\begin{aligned}
(\tilde{\beta} \otimes \mathrm{id}_{\widetilde{{}^B H}})\tilde{\beta}(b) &= ((\gamma \otimes \mathrm{id}_B)\beta \otimes \mathrm{id}_B \otimes \mathrm{id}_B)(\gamma \otimes \mathrm{id}_B)\beta(b) \\
&= (\gamma \otimes \mathrm{id}_B \otimes \mathrm{id}_B \otimes \mathrm{id}_B)((\beta \otimes \mathrm{id}_B)\gamma \otimes \mathrm{id}_B)\beta(b) \\
&= (\gamma \otimes \mathrm{id}_B \otimes \mathrm{id}_B \otimes \mathrm{id}_B)((\mathrm{id}_H \otimes \gamma)\Delta_H \otimes \mathrm{id}_B)\beta(b) \\
&= (\gamma \otimes \gamma \otimes \mathrm{id}_B)(\mathrm{id}_H \otimes \beta)\beta(b) \\
&= (\mathrm{id}_B \otimes \mathrm{id}_B \otimes (\gamma \otimes \mathrm{id}_B)\beta)(\gamma \otimes \mathrm{id}_B)\beta(b) \\
&= (\mathrm{id}_B \otimes \tilde{\Delta})\tilde{\beta}(b).
\end{aligned}
$$

Then coassociativiy follows:

$$\begin{aligned}
(\mathrm{id}_{\widetilde{^BH}} \otimes \tilde{\Delta})\tilde{\Delta} &= (\mathrm{id}_B \otimes \mathrm{id}_B \otimes \tilde{\Delta})(\mathrm{id}_B \otimes \tilde{\beta}) \\
&= \mathrm{id}_B \otimes (\mathrm{id}_B \otimes \tilde{\Delta})\tilde{\beta} \\
&= \mathrm{id}_B \otimes (\tilde{\beta} \otimes \mathrm{id}_{\widetilde{^BH}})\tilde{\beta} \\
&= (\mathrm{id}_B \otimes \tilde{\beta} \otimes \mathrm{id}_{\widetilde{^BH}})(\mathrm{id}_B \otimes \tilde{\beta}) \\
&= (\tilde{\Delta} \otimes \mathrm{id}_{\widetilde{^BH}})\tilde{\Delta}.
\end{aligned}$$

2. For $\tilde{\varepsilon}$, we start by proving it is well defined, i.e. $b_i b'_i \in \mathbb{C}1_B$ for $b_i \otimes b'_i \in \widetilde{^BH}$. Indeed, for $b_i \otimes b'_i \in \widetilde{^BH}$, we have $1_H \otimes b_i \otimes b'_i = b_{i(-1)} b'_{i(-1)} \otimes b_{i(0)} \otimes b'_{i(0)}$. Moreover $1_H \otimes b_i b'_i = b_{i(-1)} b'_{i(-1)} \otimes b_{i(0)} b'_{i(0)} = \beta(b_i b'_i)$ and hence $b_i b'_i$ is a coinvariant for β. However, as β is ergodic, this implies $b_i b'_i \in \mathbb{C}1_B$ and hence that $\tilde{\varepsilon}$ is well defined. Now we have:

$$\begin{aligned}
&(\tilde{\varepsilon} \otimes \mathrm{id}_{\widetilde{^BH}})\tilde{\Delta}(b_i \otimes b'_i) \\
&= b_i l_j(b'_{i(-1)}) \otimes r_j(b'_{i(-1)}) \otimes b'_{i(0)} \\
&= l_k(b_{i(-1)}) l_j(b'_{i(-1)}) \otimes r_j(b'_{i(-1)}) r_k(b_{i(-1)}) b_{i(0)} \otimes b'_{i(0)} \\
&= l_k(b_{i(-1)} b'_{i(-1)}) \otimes r_k(b_{i(-1)} b'_{i(-1)}) b_{i(0)} \otimes b'_{i(0)} \\
&= 1_B \otimes b_i \otimes b'_i
\end{aligned}$$

and also

$$\begin{aligned}
(\mathrm{id}_{\widetilde{^BH}} \otimes \tilde{\varepsilon})\tilde{\Delta}(b_i \otimes b'_i) &= b_i \otimes l_j(b'_{i(-1)}) \otimes r_j(b'_{i(-1)}) b'_{i(0)} \\
&= b_i \otimes b'_i \otimes 1_B.
\end{aligned}$$

which proves that $\tilde{\varepsilon}$ is indeed a co-unit.

3. For the antipode, we start by proving that the range of the linear map

$$S_0 : \widetilde{^BH} \to B \odot B : b_i \odot b'_i \mapsto l_j(b_{i(-1)}) b'_i r_j(b_{i(-1)}) \otimes b_{i(0)}$$

is contained in ${}^{coH}(B \odot B)$. We have:

$$
\begin{aligned}
&\lambda(S_0(b_i \otimes b'_i))\\
&= l_j(b_{i(-2)})_{(-1)} b'_{i(-1)} r_j(b_{i(-2)})_{(-1)} b_{i(-1)}\\
&\qquad \otimes l_j(b_{i(-2)})_{(0)} b'_{i(0)} r_j(b_{i(-2)})_{(0)} \otimes b_{i(0)}\\
&= b_{i(-3)} b'_{i(-1)} r_j(b_{i(-2)})_{(-1)} b_{i(-1)}\\
&\qquad \otimes l_j(b_{i(-2)}) b'_{i(0)} r_j(b_{i(-2)})_{(0)} \otimes b_{i(0)}\\
&= b_{i(-4)} b'_{i(-1)} S(b_{i(-2)}) b_{i(-1)} \otimes l_j(b_{i(-3)}) b'_{i(0)} r_j(b_{i(-3)}) \otimes b_{i(0)}\\
&= b_{i(-2)} b'_{i(-1)} \otimes l_j(b_{i(-1)}) b'_{i(0)} r_j(b_{i(-1)}) \otimes b_{i(0)}\\
&= 1_H \otimes S_0(b_i \otimes b'_i)
\end{aligned}
$$

and hence $\tilde{S} : \widetilde{{}^B H} \to \widetilde{{}^B H}$ is well defined. Also, we have:

$$
\begin{aligned}
&m_{\widetilde{{}^B H} \otimes \widetilde{{}^B H}}(\tilde{S} \otimes \mathrm{id}_{\widetilde{{}^B H}})(\tilde{\Delta}(b_i \otimes b'_i))\\
&= m_{\widetilde{{}^B H} \otimes \widetilde{{}^B H}}(\tilde{S}(b_i \otimes l_k(b'_{i(-1)})) \otimes r_k(b'_{i(-1)}) \otimes b'_{i(0)})\\
&= m_{\widetilde{{}^B H} \otimes \widetilde{{}^B H}}(l_j(b_{i(-1)}) l_k(b'_{i(-1)}) r_j(b_{i(-1)})\\
&\qquad \otimes b_{i(0)} \otimes r_k(b'_{i(-1)}) \otimes b'_{i(0)})\\
&= r_k(b'_{i(-1)}) l_j(b_{i(-1)}) l_k(b'_{i(-1)}) r_j(b_{i(-1)}) \otimes b_{i(0)} b'_{i(0)}
\end{aligned}
$$

and using

$$
\begin{aligned}
&b_i \otimes l_j(b'_{i(-1)}) \otimes r_j(b'_{i(-1)}) \otimes 1 \otimes b'_{(0)}\\
&= b_i \otimes l_j(b'_{i(-1)}) \otimes l_k(r_j(b'_{i(-1)})_{(-1)})\\
&\qquad \otimes r_k(r_j(b'_{i(-1)})_{(-1)}) r_j(b'_{i(-1)})_{(0)} \otimes b'_{(0)}\\
&= b_i \otimes l_j(b'_{i(-2)}) \otimes l_k(S(b'_{i(-1)})) \otimes r_k(S(b'_{i(-1)})) r_j(b'_{i(-2)}) \otimes b'_{(0)}
\end{aligned}
$$

we get

$$
\begin{aligned}
& m_{\widetilde{^BH}\otimes\widetilde{^BH}}(\tilde{S}\otimes \mathrm{id}_{\widetilde{^BH}})(\tilde{\Delta}(b_i\otimes b_i')) \\
&= r_i(b'_{i(-1)})l_j(b_{i(-1)})l_i(b'_{i(-1)})r_j(b_{i(-1)})\otimes b_{i(0)}b'_{i(0)} \\
&= l_k(S(b'_{i(-1)}))l_j(b_{i(-1)})l_m(b'_{i(-2)})r_k(S(b'_{i(-1)})) \\
&\qquad r_m(b'_{i(-2)})r_j(b_{i(-1)})\otimes b_{i(0)}b'_{i(0)} \\
&= l_k(S(b'_{i(-1)}))l_j(b_{i(-1)}b'_{i(-2)})r_k(S(b'_{i(-1)})) \\
&\qquad r_j(b_{i(-1)}b'_{i(-2)})\otimes b_{i(0)}b'_{i(0)} \\
&= l_k(S(b'_{i(-1)}))r_k(S(b'_{i(-1)}))\otimes b_ib'_{i(0)} \\
&= 1_B\otimes b_i\varepsilon_H(S(b'_{i(-1)}))b'_{i(0)} \\
&= 1_B\otimes b_ib_i' = \varepsilon_{\widetilde{^BH}}(b_i\otimes b_i')1_B\otimes 1_B
\end{aligned}
$$

where we used that

$$b_{i(-1)}b'_{i(-2)}\otimes b_{i(0)}\otimes b'_{i(-1)}\otimes b'_{i(0)} = 1\otimes b_i\otimes b'_{i(-1)}\otimes b'_{i(0)}.$$

Analogously,

$$
\begin{aligned}
& m_{\widetilde{^BH}\otimes\widetilde{^BH}}(\mathrm{id}_{\widetilde{^BH}}\otimes\tilde{S})(\tilde{\Delta}(b_i\otimes b_i')) \\
&= m_{\widetilde{^BH}\otimes\widetilde{^BH}}\Big(b_i\otimes l_j(b'_{i(-1)})\otimes \tilde{S}(r_j(b'_{i(-1)})\otimes b'_{i(0)})\Big) \\
&= m_{\widetilde{^BH}\otimes\widetilde{^BH}}\Big(b_i\otimes l_j(b'_{i(-1)})\otimes l_k(r_j(b'_{i(-1)})_{(-1)})b'_{i(0)} \\
&\qquad r_k(r_j(b'_{i(-1)})_{(-1)})\otimes r_j(b'_{i(-1)})_{(0)}\Big) \\
&= m_{\widetilde{^BH}\otimes\widetilde{^BH}}\Big(b_i\otimes l_j(b'_{i(-2)})\otimes l_k(S(b'_{i(-1)}))b'_{i(0)} \\
&\qquad r_k(S(b'_{i(-1)}))\otimes r_j(b'_{i(-2)})\Big) \\
&= l_k(S(b'_{i(-1)}))b'_{i(0)}r_k(S(b'_{i(-1)}))b_i\otimes l_j(b'_{i(-2)})r_j(b'_{i(-2)}) \\
&= l_k(S(b'_{i(-1)}))b'_{i(0)}r_k(S(b'_{i(-1)}))b_i\otimes 1_B
\end{aligned}
$$

$$\begin{aligned}
&= l_k(S(b'_{i(-1)})S(b_{i(-2)})b_{i(-1)})b'_{i(0)}r_k(S(b'_{i(-1)}) \\
&\qquad S(b_{i(-2)})b_{i(-1)})b_{i(0)} \otimes 1_B \\
&= l_k(S(b_{i(-2)}b'_{i(-1)})b_{i(-1)})b'_{i(0)}r_k(S(b_{i(-2)}b'_{i(-1)})b_{i(-1)})b_{i(0)} \otimes 1_B \\
&= l_k(b_{i(-1)})b'_i r_k(b_{i(-1)})b_{i(0)} \otimes 1_B \\
&= b_i b'_i \otimes 1_B \\
&= \varepsilon_{\widetilde{{}^BH}}(b_i \otimes b'_i)1_B \otimes 1_B.
\end{aligned}$$

With $\Delta_{\widetilde{{}^BH}}, \varepsilon_{\widetilde{{}^BH}}$ and $\tilde{S}$ as defined above, $\widetilde{{}^BH}$ is a well defined Hopf algebra.

□

Lemma 1.2.13. *$\tilde{\beta}$ is a right coaction of $\widetilde{{}^BH}$ on B, i.e.*

1. $(\tilde{\beta} \otimes \mathrm{id}_{\widetilde{{}^BH}})\tilde{\beta} = (\mathrm{id}_B \otimes \Delta_{\widetilde{{}^BH}})\tilde{\beta}$ *and*
2. $(\mathrm{id}_B \otimes \tilde{\varepsilon})\tilde{\beta} = \mathrm{id}_B$.

Proof. The first equality is already proven in the first item of the lemma 1.2.12. For the second equality, note that

$$\begin{aligned}
(\mathrm{id}_{\widetilde{{}^BH}} \otimes \tilde{\varepsilon})\tilde{\beta}(b) &= l_i(b_{-1}) \otimes \tilde{\varepsilon}(r_i(b_{(-1)}) \otimes b_{(0)}) \\
&= l_i(b_{-1}) \otimes r_i(b_{(-1)})b_{(0)} \\
&= b \otimes 1
\end{aligned}$$

proving the statement. □

Putting all the lemma's together we can prove theorem 1.2.8.

Proof of theorem 1.2.8. We will finally prove that $R_{\tilde{\beta}}$ is bijective and that $\tilde{\beta}$ and β commute.

$R_{\tilde{\beta}}$ **is bijective:** Note that the map $R_{\tilde{\beta}}$ is defined as:

$$R_{\tilde{\beta}} : B \odot B \to B \odot \widetilde{{}^BH} : b \otimes b' \mapsto (b \otimes 1_B)\tilde{\beta}(b').$$

We will prove that the map

$$R'_{\tilde{\beta}} : B \odot \widetilde{{}^{B}H} \to B \odot B : b \otimes (c_i \otimes c'_i) \mapsto (bc_i \otimes c'_i)$$

constitutes its inverse. Indeed we have:

$$\begin{aligned} R'_{\tilde{\beta}}(R_{\tilde{\beta}}(b \otimes b')) &= R'_{\tilde{\beta}}(bl_i(b'_{(-1)}) \otimes r_i(b'_{(-1)}) \otimes b'_0) \\ &= bl_i(b'_{(-1)})r_i(b'_{(-1)}) \otimes b'_{(0)} \\ &= b \otimes \varepsilon(b'_{(-1)})b'_{(0)} \\ &= b \otimes b'. \end{aligned}$$

Ot the other hand, we have:

$$\begin{aligned} & R_{\tilde{\beta}}(R'_{\tilde{\beta}}(b \otimes c_i \otimes c'_i)) \\ &= R_{\tilde{\beta}}(bc_i \otimes c'_i) \\ &= bc_i l_j(c'_{i(-1)}) \otimes r_j(c'_{i(-1)}) \otimes c'_{i(0)} \\ &= bl_j(c_{i(-1)})l_j(c'_{i(-1)}) \otimes r_j(c'_{i(-1)})r_j(c_{i(-1)})c_{i(0)} \otimes c'_{i(0)} \\ &= bl_j(c_{i(-1)}c'_{i(-1)}) \otimes r_j(c_{i(-1)}c'_{i(-1)})c_{i(0)} \otimes c'_{i(0)} \\ &= b \otimes c_i \otimes c'_i \end{aligned}$$

where we used that $c_i \otimes c'_i \in (B \odot B)^{coH}$. We may conclude that $R_{\tilde{\beta}}$ is bijective.

the actions $\tilde{\beta}$ and β commute:

$$\begin{aligned} (\beta \otimes \mathrm{id}_{\widetilde{{}^{B}H}})\tilde{\beta}(b) &= (\beta \otimes \mathrm{id}_B \otimes \mathrm{id}_B)(\gamma \otimes \mathrm{id}_B)\beta(b) \\ &= ((\beta \otimes \mathrm{id}_B)\gamma \otimes \mathrm{id}_B)\beta(b) \\ &= ((\mathrm{id}_H \otimes \gamma)\Delta_H \otimes \mathrm{id}_B)\beta(b) \\ &= (\mathrm{id}_H \otimes \gamma \otimes \mathrm{id}_B)(\mathrm{id}_H \otimes \beta)\beta(b) \\ &= (\mathrm{id}_H \otimes (\gamma \otimes \mathrm{id}_B)\beta)\beta(b) \\ &= (\mathrm{id}_H \otimes \tilde{\beta})\beta(b). \end{aligned}$$

This concludes the proof. □

Classical example. *In our classical example, let* $(x, y) \sim (g \cdot x, g \cdot y)$ *for all* $g \in G, x, y \in X$ *and define* $\tilde{G} = X \times X / \sim$ *with multiplication:*

$$[(x, y)] \star [(z, t)] = [(x, {}^{z}_{y}g \cdot t)],$$

where ${}^{z}_{y}g \cdot z = y$*. Then there is an action*

$$\alpha : X \times \tilde{G} \to X : (x, [(y, z)]) \mapsto {}^{y}_{x}g \cdot z.$$

As we have proven, $\tilde{G}$ *is isomorphic with* G *(but not in a canonical way as remarked after proposition 1.2.7).*

As B is a right $\widetilde{{}^{B}H}$-Galois object, one can wonder what the deformation $\widetilde{(\widetilde{{}^{B}H})}^{B}$ of $\widetilde{{}^{B}H}$ is.

Theorem 1.2.14. *We have* $\widetilde{(\widetilde{{}^{B}H})}^{B} \cong H$*.*

Proof. As B is a right $\widetilde{{}^{B}H}$-Galois object, we can obtain an analogous deformation of $\widetilde{{}^{B}H}$: $\widetilde{(\widetilde{{}^{B}H})}^{B} = (B \odot B)^{co(\widetilde{{}^{B}H})}$. We will now proof that $\gamma : H \to (B \odot B)^{co(\widetilde{{}^{B}H})} \subset B \odot B^{op}$ is an isomorphism. Therefore, we will first describe what the codiagonal action of $\tilde{H}$ on $B \odot B$ (which we will call $\tilde{\lambda}$) looks like. Let $b \otimes b' \in B \odot B$, then:

$$\begin{aligned}
&\tilde{\lambda}(b \otimes b') \\
&= (\mathrm{id}_B \otimes \mathrm{id}_B \otimes m_{\tilde{H}})(\mathrm{id}_B \otimes \sigma_{\tilde{H},B} \otimes \mathrm{id}_{\tilde{H}})(\tilde{\beta} \otimes \tilde{\beta})(b \otimes b') \\
&= (\mathrm{id}_B \otimes \mathrm{id}_B \otimes m_{\tilde{H}})(\mathrm{id}_B \otimes \sigma_{\tilde{H},B} \otimes \mathrm{id}_{\tilde{H}})(\gamma(b_{(-1)}) \otimes b_{(0)} \otimes \gamma(b'_{(-1)}) \otimes b'_{(0)}) \\
&= l_j(b_{(-1)}) \otimes l_k(b'_{(-1)}) \otimes r_k(b'_{(-1)}) r_j(b_{(-1)}) \otimes b_{(0)} b'_{(0)}.
\end{aligned}$$

Hence, if $b_i \otimes b'_i \in \widetilde{(\widetilde{{}^{B}H})}^{B}$, we have:

$$l_j(b_{i(-1)}) \otimes l_k(b'_{i(-1)}) \otimes r_k(b'_{i(-1)}) r_j(b_{i(-1)}) \otimes b_{i(0)} b'_{i(0)} = b_i \otimes b'_i \otimes 1_B \otimes 1_B.$$

Note furthermore, that in this case, applying $(\mathrm{id}_H \otimes m_B \otimes \mathrm{id}_B)(T_\beta \otimes \mathrm{id}_B \otimes \mathrm{id}_B)$ on both sides, we obtain:

$$\begin{aligned}
& b_{i(-1)} \otimes b_{i(0)} b'_i \otimes 1_B \\
&= l_j(b_{i(-1)})_{(-1)} \otimes l_j(b_{i(-1)})_{(0)} l_k(b'_{i(-1)}) r_k(b'_{i(-1)}) r_j(b_{i(-1)}) \otimes b_{i(0)} b'_{i(0)} \\
&= b_{i(-2)} \otimes l_j(b_{i(-1)}) r_j(b_{i(-1)}) \otimes b_{i(0)} b'_i \\
&= b_{i(-1)} \otimes 1_B \otimes b_{i(0)} b'_i,
\end{aligned}$$

proving that $T_\beta(b_i \otimes b'_i) \in H \odot \mathbb{C} \subset H \odot B$ which we will use later on.
To prove $\gamma(H) \subset (B \odot B)^{co\tilde{H}}$, we wil calculate $\tilde{\lambda}(\gamma(h))$ for $h \in H$. We have, using the calculation rules of lemma 1.2.3:

$$\begin{aligned}
& \tilde{\lambda}(\gamma(h)) \\
&= l_j(l_i(h)_{(-1)}) \otimes l_k(r_i(h)_{(-1)}) \otimes r_k(r_i(h)_{(-1)}) r_j(l_i(h)_{(-1)}) \otimes l_i(h)_{(0)} r_i(h)_{(0)} \\
&= l_j(h_{(1)}) \otimes l_k(r_i(h_{(2)})_{(-1)}) \otimes r_k(r_i(h_{(2)})_{(-1)}) r_j(h_{(1)}) \otimes l_i(h_{(2)}) r_i(h_{(2)})_{(0)} \\
&= l_j(h_{(1)}) \otimes l_k(S(h_{(3)})) \otimes r_k(S(h_{(3)})) r_j(h_{(1)}) \otimes l_i(h_{(2)}) r_i(h_{(2)}) \\
&= l_i(h_{(1)}) \otimes l_k(S(h_{(2)})) \otimes r_k(S(h_{(2)})) r_i(h_{(1)}) \otimes 1_B \\
&= l_i(h) \otimes l_k(r_i(h)_{(-1)}) \otimes r_k(r_i(h)_{(-1)}) r_i(h)_{(0)} \otimes 1_B \\
&= l_i(h) \otimes r_i(h) \otimes 1_B \otimes 1_B \\
&= \gamma(h) \otimes 1_{\tilde{H}}
\end{aligned}$$

and hence we can conclude that $\gamma(H) \subset (B \odot B)^{co\tilde{H}}$. As γ is linear and multiplicative (as we take the multiplication in $B \odot B^{op}$), it suffices to prove that γ is injective and surjective. Injectivity is direct, as $T_\beta(\gamma(h)) = h \otimes 1$. To prove surjectivity, we use the equality $b_{i(-1)} \otimes b_{i(0)} b'_i \otimes 1_B = b_{(-1)} \otimes 1_B \otimes b_{(0)} b'$ we proved above for $b_i \otimes b'_i \in \widetilde{({}^B H)}^B$. Applying a linear functional φ on B which satisfies $\varphi(1_B) = 1$ on the last tensorand, we can define $h \in H$ such that

$$T_\beta(b_i \otimes b'_i) = b_{i(-1)} \otimes b_{i(0)} b'_i = b_{(-1)} \varphi(b_{(0)} b') \otimes 1_B = h \otimes 1.$$

Then $\gamma(h) = b_i \otimes b'_i$ which proves γ is an algebra isomorphism.

Now note that $\tilde{\gamma} : \widetilde{{}^B H} \to B \odot B$ associated to the Galois map $R_{\tilde{\beta}}$ is in fact the identity. Indeed if $b_i \otimes b'_i \in \widetilde{{}^B H}$, then $b_{i(-1)} b'_{i(-1)} \otimes b_{i(0)} \otimes b'_{i(0)} = b'_i \otimes b_i \otimes 1$ and hence

$$\begin{aligned}
R_{\tilde{\beta}}(b_i \otimes b'_i) &= b_i l_j(b'_{i(-1)}) \otimes r_j(b'_{i(-1)}) \otimes b'_{i(0)} \\
&= l_k(b_{i(-1)}) l_j(b'_{i(-1)}) \otimes r_j(b'_{i(-1)}) r_k(b_{i(-1)}) b_{i(0)} \otimes b'_{i(0)} \\
&= l_k(b_{i(-1)} b'_{i(-1)}) \otimes r_j(b_{i(-1)} b'_{i(-1)}) b_{i(0)} \otimes b'_{i(0)} \\
&= 1 \otimes b_i \otimes b'_i.
\end{aligned}$$

Hence for $b_i \otimes b'_i \in (B \odot B)^{co(\widetilde{{}^B H})}$ the coproduct reduces to $\Delta_{(B\odot B)^{co(\widetilde{{}^B H})}}(b_i \otimes b'_i) = \tilde{\beta}(b_i) \otimes b'_i = l_j(b_{i(-1)}) \otimes r_j(b_{i(-1)}) \otimes b_{i(0)} \otimes b'_i$. Therefore, for $b_i \otimes b'_i = l_i(h) \otimes r_i(h)$, $h \in H$, one has

$$\begin{aligned}
\Delta_{(B\odot B)^{co(\widetilde{{}^B H})}}(\gamma(h)) &= l_j(l_i(h)_{(-1)}) \otimes r_j(l_i(h)_{(-1)}) \otimes l_i(h)_{(0)} \otimes r_i(h) \\
&= l_j(h_{(1)}) \otimes r_j(h_{(1)}) \otimes l_i(h_{(2)}) \otimes r_i(h_{(2)}) \\
&= (\gamma \otimes \gamma)\Delta(h).
\end{aligned}$$

Moreover, one has $\varepsilon_{\widetilde{{}^B H^B}}(\gamma(h)) = l_i(h) r_i(h) = \varepsilon(h)$. This proves that γ is an isomorphism of bialgebras. As the antipode of a Hopf algebra is unique given the coproduct and counit, the proof is complete. $\square$

With this deformation procedure at hand, we look at the consequences: what does it mean for a Hopf algebra to be a deformation of another Hopf algebra? And what can we say about the structure on the set of Hopf algebras and bi-Galois objects? This is the topic of the next section.

1.3 The Harrison Groupoid and Hopf-Galois equivalence

In the second section, we described how to deform a Hopf algebra H, given an H-Galois object. In this section, we investigate what structure this imposes on the set of Hopf algebras and on the set of bi-Galois objects. It turns out that the Hopf-Galois deformations induce the structure of a groupoid on the set of bi-Galois objects. Equivalently defining a category with as objects Hopf algebras and morphisms bi-Galois objects, this category is a groupoid in the categorical meaning. It follows that the Hopf-Galois deformation defined in the previous paragraph is indeed a deformation (i.e. it is reversible). Will we do this in these second subsection. First we have a look at deformations of comodules: what is the relation between the comodules of a Hopf algebra H and those of a deformation of H?

1.3.1 Hopf-Galois deformation of H-comodules

In this subsection, we have a closer look to the comodules of a Hopf algebra. We will prove that having a deformation of a Hopf algebra H to a Hopf algebra $\tilde{H}$ induces a deformation of its comudule(-algebra)s to $\tilde{H}$-comodule(-algebra)s. Moreover, this deformation turns out to be categorical: morphisms can be deformed as well and the deformation is compatible with the monoidal structure. We use the results of [90] and [91], found in [87].

Proposition 1.3.1. *Let H be a Hopf algebra, B a left H-Galois object and $\tilde{H}$ the Hopf-Galois deformation of H along B. Suppose furthermore that V is a unital right H-comodule with coaction $\alpha : V \to V \odot H$. Then*

$$\tilde{V} = V \underset{H}{\boxdot} B := \{z \in V \odot B \,|(\alpha \otimes \mathrm{id}_B)(z) = (\mathrm{id}_V \otimes \beta)(z)\}$$

defines a right $\tilde{H}$-comodule with coaction

$$\tilde{\alpha} : \tilde{V} \to \tilde{V} \odot \tilde{H} : z \mapsto (\mathrm{id}_V \otimes \tilde{\beta})(z).$$

We call $\tilde{V}$ the Hopf-Galois deformation of V along B.

Furthermore, if V is a unital right H-comodule-algebra with a coaction α, then also $\tilde{V}$ will be a $\tilde{H}$-comodule-algebra with coaction $\tilde{\alpha}$.

Remark 1.3.2. *In the future, we will use the notation V for an H-comodule and A for an H-comodule-algebra.*

Proof of proposition 1.3.1. It is clear that $\tilde{V}$ is a vector space. To prove that $\tilde{\alpha}$ is well-defined, let $z \in \tilde{V}$.

Then

$$
\begin{aligned}
(\alpha \otimes \mathrm{id}_B \otimes \mathrm{id}_{\tilde{H}})(\mathrm{id}_V \otimes \tilde{\beta})(z) &= (\alpha \otimes \tilde{\beta})(z) \\
&= (\mathrm{id}_V \otimes \mathrm{id}_H \otimes \tilde{\beta})(\alpha \otimes \mathrm{id}_B)(z) \\
&= (\mathrm{id}_V \otimes \mathrm{id}_H \otimes (\gamma \otimes \mathrm{id}_B)\beta)(\mathrm{id}_V \otimes \beta)(z) \\
&= (\mathrm{id}_V \otimes (\mathrm{id}_H \otimes \gamma)\Delta_H \otimes \mathrm{id}_B)(\mathrm{id}_V \otimes \beta)(z) \\
&= (\mathrm{id}_V \otimes (\beta \otimes \mathrm{id}_B)\gamma \otimes \mathrm{id}_B)(\mathrm{id}_V \otimes \beta)(z) \\
&= (\mathrm{id}_V \otimes \beta \otimes \mathrm{id}_{\tilde{H}})(\mathrm{id}_V \otimes \tilde{\beta})(z) \\
&= (\mathrm{id}_V \otimes \beta \otimes \mathrm{id}_{\tilde{H}})(\tilde{\alpha}(z)).
\end{aligned}
$$

Hence $\tilde{\alpha}$ is well defined. Also, as $\tilde{\beta}$ is a right coaction of $\tilde{H}$ on B, it is straightforward to prove that $\tilde{\alpha}$ is one for $\tilde{V}$. Analogously, if V is a comodule-algebra, $\tilde{\alpha}$ is multiplicative as $\tilde{\beta}$ is. $\square$

Classical example. *Suppose G is a group with left action on X such that (G, X) is a left Galois pair. Now suppose Y is a space with a right action of G on it. Then define the following equivalence relation in $Y \times X : (y \cdot g, x) \sim (y, g \cdot x)$. We call $\tilde{Y} = Y \underset{G}{\square} X = Y \times X/\sim$ the Hopf-Galois deformation of Y along X. The right action of $\tilde{G}$ on $\tilde{Y}$ is*

$$\tilde{Y} \times \tilde{G} \to \tilde{Y} : \big([(y, x)], [(z, t)]\big) \mapsto [(y, {}^{z}_{x}g \cdot t)].$$

Note that, in this case $\tilde{Y}$ is isomorphic to Y. Indeed,choose an arbitrary $x_0 \in X$, then we have the map

$$\varphi : Y \to \tilde{Y} : y \mapsto [(y, x_0)].$$

There is even more: the deformation of comodules described above is compatible with taking tensor products of comodules.

Proposition 1.3.3. *Let H be a Hopf algebra, B a $(H$-$\tilde{H})$-bi-Galois object and V_1, V_2 right H-comodules with respective coactions α_1 and α_2.*

- *Consider the base field k of H with trivial (right) comodule structure. Then there exists an isomorphism $\varphi_1 : k \underset{H}{\square} B \xrightarrow{\cong} k$ of right $\tilde{H}$-comodules.*

- *There is a linear isomorphism*

  $$\varphi_2 : (V_1 \underset{H}{\boxdot} B) \odot (V_2 \underset{H}{\boxdot} B) \to (V_1 \odot V_2) \underset{H}{\boxdot} B : v_i \otimes b_i \otimes w_i \otimes b'_i \mapsto v_i \otimes w_i \otimes b_i b'_i$$

  (summation is implied) of right $\tilde{H}$-comodules.

- *If $\varphi_3 : V_1 \to V_2$ is a morphism of right H-comodules, then $\varphi'_3 = \varphi_3 \otimes \mathrm{id}_B$: $V_1 \underset{H}{\boxdot} B \to V_2 \underset{H}{\boxdot} B$ is a morphism of right $\tilde{H}$-comodules.*

Proof. • Denote with $\alpha_{\mathbb{C}}$ the trivial coaction of H on k and let $\sum_i \lambda_i \otimes b_i$ be an arbitrary element in $k \underset{H}{\square} B$. As $\sum_i \lambda_i \otimes b_i = \sum_i 1_{\mathbb{C}} \otimes \lambda_i b_i = 1 \otimes b$ for some $b \in B$, one has $1_{\mathbb{C}} \otimes 1_H \otimes b = \alpha_{\mathbb{C}}(1) \otimes b = 1_{\mathbb{C}} \otimes \beta(b)$ and as β is ergodic, $b = \lambda 1_B$ with $\lambda \in \mathbb{C}$. We can conclude that $\varphi_1(\sum \lambda_i \otimes b_i) = \lambda$ constitutes an isomorphism $k \underset{H}{\square} B \cong k$.

- First we will prove that φ_2 is well defined. Note that if $v_i \otimes b_i \otimes w_i \otimes b'_i \in (V_1 \underset{H}{\boxdot} B) \odot (V_2 \underset{H}{\boxdot} B)$, then

  $$\begin{aligned} v_i \otimes w_i \otimes \beta(b_i b'_i) &= v_i \otimes w_i \otimes b_{i(-1)} b'_{i(-1)} \otimes b_{i(0)} b'_{i(0)} \\ &= v_{i(0)} \otimes w_i \otimes v_{i(1)} b'_{i(-1)} \otimes b_i b'_{i(0)} \\ &= v_{i(0)} \otimes w_{i(0)} \otimes v_{i(1)} w_{i(1)} \otimes b_i b'_i \\ &= \alpha_{V_1 \odot V_2}(v_i \otimes w_i) \otimes b_i b'_i, \end{aligned}$$

  so φ_2 is well defined. Moreover, if $v_i \otimes w_i \otimes b_i \in (V_1 \odot V_2) \underset{H}{\boxdot} B$, then $v_{i(0)} \otimes l_j(v_{i(1)}) \otimes w_i \otimes r_j(v_{i(1)}) b_i \in V_1 \underset{H}{\boxdot} B \odot V_2 \underset{H}{\boxdot} B$. Indeed,

  $$\begin{aligned} & v_{i(0)} \otimes \beta(l_j(v_{i(1)})) \otimes w_i \otimes r_j(v_{i(1)}) b_i \\ &= v_{i(0)} \otimes l_j(v_{i(1)})_{(-1)} \otimes l_j(v_{i(1)})_{(0)} \otimes w_i \otimes r_j(v_{i(1)}) b_i \\ &= v_{i(0)} \otimes v_{i(1)} \otimes l_j(v_{i(2)}) \otimes w_i \otimes r_j(v_{i(2)}) b_i \\ &= \alpha_1(v_{i(0)}) \otimes l_j(v_{i(1)}) \otimes w_i \otimes r_j(v_{i(1)}) b_i \end{aligned}$$

and

$$
\begin{aligned}
& v_{i(0)} \otimes l_j(v_{i(1)}) \otimes w_i \otimes \beta(r_j(v_{i(1)})b_i) \\
&= v_{i(0)} \otimes l_j(v_{i(1)}) \otimes w_i \otimes r_j(v_{i(1)})_{(-1)} b_{i(-1)} \otimes r_j(v_{i(1)})_{(0)} b_{i(0)} \\
&= v_{i(0)} \otimes l_j(v_{i(1)}) \otimes w_i \otimes S(v_{i(2)}) b_{i(-1)} \otimes r_j(v_{i(1)}) b_{i(0)} \\
&= v_{i(0)} \otimes l_j(v_{i(1)}) \otimes w_{i(0)} \otimes S(v_{i(2)}) v_{i(3)} w_{i(1)} \otimes r_j(v_{i(1)}) b_i \\
&= v_{i(0)} \otimes l_j(v_{i(1)}) \otimes w_{i(0)} \otimes w_{i(1)} \otimes r_j(v_{i(1)}) b_i \\
&= v_i \otimes l_j(v_{i(1)}) \otimes \alpha_2(w_{i(0)}) \otimes r_j(v_{i(1)}) b_i .
\end{aligned}
$$

Hence we can define

$$
\varphi_2' : (V_1 \odot V_2) \underset{H}{\boxdot} B \to (V_1 \underset{H}{\boxdot} B) \odot (V_2 \underset{H}{\boxdot} B) :
$$

$$
v_i \otimes w_i \otimes b_i \mapsto v_{i(0)} \otimes l_j(v_{i(1)}) \otimes w_i \otimes r_j(v_{i(1)}) b_i ,
$$

and then we have for $v_i \otimes w_i \otimes b_i \in (V_1 \odot V_2) \underset{H}{\boxdot} B$

$$
\begin{aligned}
\varphi_2(\varphi_2'(v_i \otimes w_i \otimes b_i)) &= \varphi_2\Big(v_{i(0)} \otimes l_j(v_{i(1)}) \otimes w_i \otimes r_j(v_{i(1)}) b_i\Big) \\
&= v_{i(0)} \otimes w_i \otimes l_j(v_{i(1)}) r_j(v_{i(1)}) b_i \\
&= v_i \otimes w_i \otimes b_i
\end{aligned}
$$

and for $v_i \otimes b_i \otimes w_i \otimes b_i' \in (V_1 \underset{H}{\boxdot} B) \odot (V_2 \underset{H}{\boxdot} B)$,

$$
\begin{aligned}
\varphi_2'(\varphi_2(v_i \otimes b_i \otimes w_i \otimes b_i')) &= \varphi_2'(v_i \otimes w_i \otimes b_i b_i') \\
&= v_{i(0)} \otimes l_j(v_{i(1)}) \otimes w_i \otimes r_j(v_{i(1)}) b_i b_i' \\
&= v_i \otimes l_j(b_{i(-1)}) \otimes w_i \otimes r_j(b_{i(-1)}) b_{i(0)} b_i' \\
&= v_i \otimes b_i \otimes w_i \otimes b_i'
\end{aligned}
$$

proving that φ_2 is a bijection. Finally note that $(V_1 \underset{H}{\boxdot} B) \odot (V_2 \underset{H}{\boxdot} B)$ has the structure of $\tilde{H}$-comodule via the codiagonal coaction $\lambda_{\tilde{\beta}}$. With this

$\tilde{H}$-comodule structure, we have for $v_i \otimes b_i \otimes w_i \otimes b'_i \in (V_1 \underset{H}{\boxdot} B) \odot (V_2 \underset{H}{\boxdot} B)$,

$$\begin{aligned}
&(\mathrm{id}_{V_1 \odot V_2} \otimes \tilde{\beta})(\varphi_2(v_i \otimes b_i \otimes w_i \otimes b'_i)) \\
&= v_i \otimes w_i \otimes \tilde{\beta}(b_i b'_i) \\
&= v_i \otimes w_i \otimes b_{i[0]} b'_{i[0]} \otimes b_{i[1]} b'_{i[1]} \\
&= (\varphi_2 \otimes \mathrm{id}_{\tilde{H}})(v_i \otimes b_{i[0]} \otimes w_i \otimes b'_{i[0]} \otimes b_{i[1]} b'_{i[1]}) \\
&= (\varphi_2 \otimes \mathrm{id})(\lambda_{\tilde{\beta}}(v_i \otimes b_i \otimes w_i \otimes b'_i))
\end{aligned}$$

where $\tilde{\beta}(b) = b_{[0]} \otimes b_{[1]}$ is the Sweedler notation for the right $\tilde{H}$-coaction. Note that the composition of Sweedler notations for β_1 and β_2 is correct as the two coactions commute.This proves that φ_2 is an isomorphism of $\tilde{H}$-comodules.

- For $v_i \otimes b_i \in V_1 \underset{H}{\boxdot} B$, we have

$$\begin{aligned}
(\alpha_2 \otimes \mathrm{id}_B)(\varphi_3(v_i) \otimes b_i) &= (\varphi_3 \otimes \mathrm{id}_H)(\alpha_1(v_i) \otimes b_i) \\
&= \varphi_3(v_i) \otimes \beta(b_i) \\
&= (\mathrm{id} \otimes \beta)(\varphi_3(v_i) \otimes b_i)
\end{aligned}$$

and hence φ'_3 is a well defined linear map. Moreover, by construction,

$$\begin{aligned}
(\varphi'_3 \otimes \mathrm{id}_{\tilde{H}})(\mathrm{id}_{V_1} \otimes \tilde{\beta}) &= (\varphi_3 \otimes \mathrm{id}_B \otimes \mathrm{id}_{\tilde{H}})(\mathrm{id}_{V_1} \otimes \tilde{\beta}) \\
&= (\mathrm{id}_{V_2} \otimes \tilde{\beta})(\varphi_3 \otimes \mathrm{id}_B) \\
&= (\mathrm{id}_{V_2} \otimes \tilde{\beta})\varphi'_3
\end{aligned}$$

proving φ'_3 is a morphism of right $\tilde{H}$ comodules. □

Proposition 1.3.4. *Let H be a Hopf algebra, B a $(H$-$\tilde{H})$-bi-Galois object and V a finite dimensional right H-comodule with coaction α. Then $V \underset{H}{\boxdot} B$ is finite dimensional as well.*

Proof. For the proof of this proposition, we use the following characterization of finite dimensional vector spaces and H-comodules, found in [26], [63], [57]. A vector space V is finite dimensional if and only if there exists a vector space W and linear maps $e : W \odot V \to k$ and $d : k \to V \odot W$ such that $\mathrm{id}_V = (\mathrm{id}_V \otimes e)(d \otimes \mathrm{id}_V)$ and $\mathrm{id}_W = (e \otimes \mathrm{id}_W)(\mathrm{id}_W \otimes d)$. If V is a finite dimensional comodule, the dual comodule V^* is the vector space W and the maps d and e are H-colinear.

So, let V be a finite dimensional H-comodule. Let W, d and e as above. Define then

$$e' : (W \underset{H}{\boxdot} B) \odot (V \underset{H}{\boxdot} B) \stackrel{\varphi_2}{\to} (W \odot V) \underset{H}{\boxdot} B \stackrel{e \otimes \mathrm{id}_B}{\to} k :$$

$$(w_i \otimes b_i) \otimes (v_i \otimes b_i') \mapsto e(w_i \otimes v_i) b_i b_i'$$

$$d' : k \stackrel{d \otimes \mathrm{id}_B}{\to} (V \odot W) \underset{H}{\boxdot} B \stackrel{\varphi_2^{-1}}{\to} (V \underset{H}{\boxdot} B) \odot (W \underset{H}{\boxdot} B) :$$

$$1 \mapsto v_{i(0)} \otimes l_j(v_{i(1)}) \otimes w_i \otimes r_j(v_{i(1)}).$$

where $d(1) = v_i \otimes w_i$ in the definition of d'. Then we have

$$\begin{aligned}
&(\mathrm{id}_{V \underset{H}{\boxdot} B} \otimes e')(d' \otimes \mathrm{id}_{V \underset{H}{\boxdot} B})(v_k' \otimes b_k) \\
&= (\mathrm{id}_{V \underset{H}{\boxdot} B} \otimes e')\Big(v_{i(0)} \otimes l_j(v_{i(1)}) \otimes w_i \otimes r_j(v_{i(1)}) \otimes v_k' \otimes b_k\Big) \\
&= v_{i(0)} \otimes l_j(v_{i(1)}) e(w_i \otimes v_k') r_j(v_{i(1)}) b_k \\
&= e(w_i \otimes v_k') v_i \otimes b_k \\
&= v_k' \otimes b_k
\end{aligned}$$

as $e(w_i \otimes v')v_i = v'$ for every $v' \in V$. Analogously

$$\begin{aligned}
&(e' \otimes \mathrm{id}_{W \underset{H}{\boxdot} B})(\mathrm{id}_{W \underset{H}{\boxdot} B} \otimes d')(w_k' \otimes b_k) \\
&= (e' \otimes \mathrm{id}_{W \underset{H}{\boxdot} B})\Big(w_k' \otimes b_k \otimes v_{i(0)} \otimes l_j(v_{i(1)}) \otimes w_i \otimes r_j(v_{i(1)})\Big) \\
&= \Big(e(w_k' \otimes v_{i(0)}) b_k l_j(v_{i(1)})\Big) w_i \otimes r_j(v_{i(1)}) \\
&= w_i \otimes e(w_k' \otimes v_{i(0)}) b_k l_j(v_{i(1)}) r_j(v_{i(1)})
\end{aligned}$$

$$
\begin{aligned}
&= e(w'_k \otimes v_i)w_i \otimes b_k \\
&= w'_k \otimes b_k
\end{aligned}
$$

as $e(w' \otimes v_i)w_i = w'$ for every $w' \in W$. This concludes the proof. □

The properties of propositions 1.3.3 and 1.3.4 can be summarized elegantly in the language of categories. We will do this at the end of subsection 1.3.2.

1.3.2 Groupoid structure on the set of bi-Galois objects

In this subsection, we prove that we can make a groupoid of the set of bi-Galois objects. Alternatively, in the categorical sense, the category with as objects Hopf algebras and morphisms the bi-Galois objects, is a categorical groupoid. This result is stated in theorem 1.3.8 and originates from [87].

Proposition 1.3.5. *Let H_1, H_2 and H_3 be Hopf algebras; $(B_1, \beta_1, \tilde{\beta}_1)$ a (H_1-H_2)-bi-Galois object and $(B_2, \beta_2, \tilde{\beta}_2)$ a (H_2-H_3)-bi-Galois object. Then $(B_1 \underset{H_2}{\boxdot} B_2, \beta_1 \otimes \mathrm{id}_{B_2}, \mathrm{id}_{B_1} \otimes \tilde{\beta}_2)$ with*

$$B_1 \underset{H_2}{\boxdot} B_2 = \{z \in B_1 \odot B_2 | (\tilde{\beta}_1 \otimes \mathrm{id}_{B_2})(z) = (\mathrm{id}_{B_1} \otimes \beta_2)(z)\}$$

is a (H_1-H_3)-bi-Galois object. Moreover, $\boxdot$ defines a associative operation on the set of bi-Galois objects.

Proof. With proposition 1.3.3 at hand, it is easy to prove the result. Note first that the linear bijection T_{β_1} is in fact an isomorphism of H_2-comodules where on $B_1 \odot B_1$ one has the codiagonal action and on $H_1 \odot B_1$, the coaction of H_2 on the second tensorand. Indeed one has for $b, b' \in B_1$

$$
\begin{aligned}
(T_{\beta_1} \otimes \mathrm{id}_{\tilde{H}})\lambda_{\beta_2}(b_i \otimes b'_i) &= (T_{\beta_1} \otimes \mathrm{id}_{\tilde{H}})(b_{i[0]} \otimes b'_{i[0]} \otimes b_{i[1]} b'_{i[1]}) \\
&= b_{i(-1)} \otimes b_{i[0]} b'_{i[0]} \otimes b_{i[1]} b'_{i[1]} \\
&= b_{i(-1)} \otimes \beta_2(b_{i(0)} b'_i) \\
&= (\mathrm{id} \otimes \beta_2) T_{\beta_1}(b_i \otimes b'_i)
\end{aligned}
$$

where $\beta_2(b) = b_{[0]} \otimes b_{[1]}$ for $b \in B_1$.

Then, using the linear isomorphism $\varphi : (B_1 \underset{H}{\boxdot} B_2) \odot (B_1 \underset{H}{\boxdot} B_2) \to (B_1 \odot B_1) \underset{H}{\boxdot} B_2$ as in proposition 1.3.3, one has the following composition of linear bijections:

$$\begin{array}{rcl}
(B_1 \underset{H_2}{\boxdot} B_2) \odot (B_1 \underset{H_2}{\boxdot} B_2) & \overset{\varphi_2}{\to} & (B_1 \odot B_1) \underset{H_2}{\boxdot} B_2 \\
& \overset{T_{\beta_1} \otimes \mathrm{id}_{B_2}}{\to} & (H_1 \odot B_1) \underset{H_2}{\boxdot} B_2 = H_1 \odot (B_1 \underset{H_2}{\boxdot} B_2) \\
(b_i \otimes c_i) \otimes (b'_i \otimes c'_i) & \mapsto & b_{i(-1)} \otimes b_{i(0)} b'_i \otimes c_i c'_i
\end{array}$$

proving that $(B_1 \underset{H_2}{\boxdot} B_2)$ is a left H_1-Galois object. Analogously

$$\begin{array}{rcl}
(B_1 \underset{H_2}{\boxdot} B_2) \odot (B_1 \underset{H_2}{\boxdot} B_2) & \to & (B_1 \underset{H_2}{\boxdot} B_2) \odot H_3 \\
(b_i \otimes c_i) \otimes (b'_i \otimes c'_i) & \mapsto & b_i b'_i \otimes c_i c'_{i[0]} \otimes c'_{i[1]}
\end{array}$$

(where $\tilde{\beta}_2(c) = c_{[0]} \otimes c_{[1]}$ for $c \in B_2$) is a linear bijection proving that $(B_1 \underset{H_2}{\boxdot} B_2)$ is a right H_3-Galois object.

Moreover, if H_4 is a Hopf algebra and B_3 is a (H_3-H_4)-bi-Galois object, we have

$$\begin{array}{ll}
\multicolumn{2}{l}{B_1 \underset{H_2}{\boxdot} (B_2 \underset{H_3}{\boxdot} B_3)} \\
= & \{z \in B_1 \odot (B_2 \underset{H_3}{\boxdot} B_3) | (\tilde{\beta}_1 \otimes \mathrm{id}_{B_2} \otimes \mathrm{id}_{B_3})(z) = (\mathrm{id}_{B_1} \otimes \beta_2 \otimes \mathrm{id}_{B_3})(z)\} \\
= & \begin{array}{l} \{z \in B_1 \odot B_2 \odot B_3 | (\tilde{\beta}_1 \otimes \mathrm{id}_{B_2} \otimes \mathrm{id}_{B_3})(z) = (\mathrm{id}_{B_1} \otimes \beta_2 \otimes \mathrm{id}_{B_3})(z) \\ \qquad \wedge (\mathrm{id}_{B_1} \otimes \tilde{\beta}_2 \otimes \mathrm{id}_{B_3})(z) = (\mathrm{id}_{B_1} \otimes \mathrm{id}_{B_2} \otimes \beta_3)(z)\} \end{array} \\
= & \{z \in (B_1 \underset{H_2}{\boxdot} B_2) \odot B_3 | (\mathrm{id}_{B_1} \otimes \tilde{\beta}_2 \otimes \mathrm{id}_{B_3})(z) = (\mathrm{id}_{B_1} \otimes \mathrm{id}_{B_2} \otimes \beta_3)(z)\} \\
= & (B_1 \underset{H_2}{\boxdot} B_2) \underset{H_3}{\boxdot} B_3
\end{array}$$

proving that $\boxdot$ is an associative operation. $\square$

Proposition 1.3.6. *Let H_1, H_2 be Hopf algebras and B a (H_1-H_2)-bi-Galois object. Then*

1. *(H_1, Δ, Δ) is a (H_1-H_1)-bi-Galois object and hence $\widetilde{{}^{H_1}H_1} \cong H_1$.*

2. *$H_1 \underset{H_1}{\boxdot} B \cong B$ and $B \underset{H_2}{\boxdot} H_2 \cong B$ as (H_1-H_2)-bi-Galois objects..*

3. *Define*
$$B' := (H_2 \odot B)^{coH_2} \subset H_2 \odot B^{op}.$$
Then $(B', \Delta \otimes \mathrm{id}_B)$ is a (H_2-H_1)-bi-Galois object such that $(B \underset{H_2}{\boxdot} B') \cong H_1$ as (H_1-H_1)-bi-Galois objects and $(B' \underset{H_1}{\boxdot} B) \cong H_2$ as (H_2-H_2)-bi-Galois objects.

Proof. 1. We proved already after definition 1.2.1 that (H, Δ) is a left and right H-Galois object for every Hopf algebra H. Obviously, the two actions commute by coassociativity.

2. It can easily be seen that $\beta_1 : B \to H_1 \underset{H_1}{\boxdot} B$ and $\varepsilon_1 \otimes \mathrm{id}_B : H_1 \underset{H_1}{\boxdot} B \to B$ (resp. $\beta_2 : B \to B \underset{H_2}{\boxdot} H_2$ and $\mathrm{id} \otimes \varepsilon_2 : B \underset{H_2}{\boxdot} H_2 \to B$) are mutually inverse isomorphisms of left H_1- and right H_2-comodule-algebras.

3. We give a sketch of the proof. Details can be found in [88]. On $B' = (H_2 \odot B)^{coH_2}$ we put the left H_2-comodule structure given by the coproduct on H_2. For the right H_1-comodule structure, we claim that for $h_i \otimes b_i \in (H_2 \odot B)^{coH_2}$, $h_{i(1)} \otimes h_{i(2)} \otimes b_i \in (H_2 \odot B)^{coH_2} \odot (B \odot B)^{coH_2}$ and hence $(\mathrm{id}_{H_2} \otimes \mathrm{id}_B \otimes \gamma^{-1})(h_{i(1)} \otimes h_{i(2)} \otimes b_i) \in (H_2 \odot B)^{coH_2} \odot H_1$. We define this coaction by $\delta_1 : B' \to B' \odot H_1$. Furthermore $B \underset{H_2}{\boxdot} B'$ is a (H_1-H_1)-bi-Galois object.

Finally, we claim to have algebra isomorphisms
$$B \underset{H_2}{\boxdot} (H_2 \odot B)^{coH_2} \overset{\mathrm{id}_B \otimes \varepsilon_2 \otimes \mathrm{id}_B}{\cong} (B \odot B)^{coH_2} \overset{\gamma^{-1}}{\cong} H_1$$
where the inverse of the first is $\beta_2 \otimes \mathrm{id}_B$ and where we used theorem 1.2.14 for the last isomorphism. Moreover, it is easy to see that $(\mathrm{id}_H \otimes \mathrm{id}_B \otimes \varepsilon_2 \otimes \mathrm{id}_B)(\beta_1 \otimes \mathrm{id}_{B'}) = (\beta_1 \otimes \mathrm{id}_B)(\mathrm{id}_B \otimes \varepsilon_2 \otimes \mathrm{id}_B)$ and by lemma 1.2.3, $\Delta \circ \gamma^{-1} = (\mathrm{id} \otimes \gamma^{-1})(\beta_1 \otimes \mathrm{id})$. Hence H_1 and $B \underset{H_2}{\boxdot} (H_2 \odot B)^{coH_2}$ are isomorphic as left H_1-Galois objects. Now note that

$$\begin{aligned}
&(\beta_2 \otimes \mathrm{id}_B \otimes \mathrm{id}_{H_1})(\gamma \otimes \mathrm{id}_{H_1})\Delta_1(h) \\
&= \beta_2(l_i(h_{(1)})) \otimes r_i(h_{(1)}) \otimes h_{(2)} \\
&= l_j((l_i(h_{(1)})_{(-1)})) \otimes r_j((l_i(h_{(1)})_{(-1)})) \otimes l_i(h_{(1)})_{(0)} \otimes r_i(h_{(1)}) \otimes h_{(2)} \\
&= l_j(h_{(1)}) \otimes r_j(h_{(1)}) \otimes l_i(h_{(2)}) \otimes r_i(h_{(2)}) \otimes h_{(3)}
\end{aligned}$$

and also

$$
\begin{aligned}
&(\mathrm{id}_B \otimes \delta_1)(\beta_2 \otimes \mathrm{id}_B)\gamma(h) \\
&= (\mathrm{id}_B \otimes \delta_1)(\beta_2(l_i(h)) \otimes r_i(h)) \\
&= l_j(h_{(1)}) \otimes \delta_1\big(r_j(h_{(1)}) \otimes l_i(h_{(2)}) \otimes r_i(h_{(2)})\big) \\
&= l_j(h_{(1)}) \otimes r_j(h_{(1)}) \otimes (\mathrm{id}_B \otimes \mathrm{id}_B \otimes \gamma^{-1})\Big(\beta_2\big(l_i(h_{(2)})\big) \otimes r_i(h_{(2)})\Big) \\
&= l_j(h_{(1)}) \otimes r_j(h_{(1)}) \otimes l_i(h_{(2)}) \otimes r_i(h_{(2)}) \otimes \gamma^{-1}\big(l_k(h_{(3)}) \otimes r_k(h_{(3)})\big) \\
&= l_j(h_{(1)}) \otimes r_j(h_{(1)}) \otimes l_i(h_{(2)}) \otimes r_i(h_{(2)}) \otimes h_{(3)}.
\end{aligned}
$$

This proves that H_1 and $B \underset{H_2}{\boxdot} (H_2 \odot B)^{coH_2}$ are also isomorphic as right H_1-Galois objects and hence $(B \underset{H_2}{\boxdot} B') \cong H_1$ as $(H_1\text{-}H_1)$-bi-Galois objects. Finally, analogously, for B' there exists a $(H_1\text{-}H_2)$-bi-Galois object B'' such that $(B' \underset{H_1}{\boxdot} B'') \cong H_2$ as $(H_2\text{-}H_2)$-bi-Galois objects. It suffices now to prove that $B'' \cong B$. We have

$$
B \cong B \underset{H_2}{\boxdot} H_2 \cong B \underset{H_2}{\boxdot} B' \underset{H_1}{\boxdot} B'' \cong H_1 \underset{H_1}{\boxdot} B'' \cong B''
$$

proving the claim.

□

Corollary 1.3.7. *Let H be a Hopf algebra and B a left H-Galois object. Then*

$$
\widetilde{{}^{B}H} \cong \widetilde{H^{B'}}.
$$

Combining the previous results, one has the following theorem.

Theorem 1.3.8 ([87]). *The set $BiGalois$ of isomorphism classes of bi-Galois objects is a groupoid with composition $\boxdot$. Equivalently, if $\mathbf{H}$ is the category with as objects the Hopf algebras and morphisms the isomorphism classes of bi-Galois objects with as composition the operation $\boxdot$, then $\mathbf{H}$ is a categorical groupoid.*

One can summarize the properties of propositions 1.3.3 and 1.3.4 in a categorical way.

Definition 1.3.9. *A strict monoidal category is a category $\mathcal{C}$ equipped with a bifunctor $\bigcirc : \mathcal{C} \bigcirc \mathcal{C} \to \mathcal{C}$ called the tensor product with a unit object I such that $(A \bigcirc B) \bigcirc C = A \bigcirc (B \bigcirc C)$ and $A \bigcirc I = I \bigcirc A$.*

Proposition 1.3.10. *The set of (right) comodules (resp. finite dimensional comodules) of a Hopf algebra H form a strict monoidal category $Comod(H)$ (resp. $Comod_f(H)$).*

With this at hand, we state a result of Ulbrich [90, 91], also found in [87] and explained in [26].

Theorem 1.3.11 ([90, 91]). *Let H and $\tilde{H}$ be Hopf algebras. Then the following are equivalent:*

1. *There exists a (H-$\tilde{H}$)-bi-Galois object B.*
2. *There is an equivalence of strict monoidal categories $\mathcal{F} : Comod(H) \overset{\cong}{\to} Comod(\tilde{H})$.*

In this case

$$\mathcal{F} : Comod(H) \overset{\cong}{\to} Comod(\tilde{H}) : \begin{cases} V \mapsto V \underset{H}{\boxdot} B \\ \left(\varphi : V_1 \to V_2\right) \mapsto \left(\varphi \otimes \mathrm{id}_B : V_1 \underset{H}{\boxdot} B \to V_2 \underset{H}{\boxdot} B\right). \end{cases}$$

and also $Comod_f(H) \overset{\cong}{\to} Comod_f(\tilde{H})$.

Proof. The implication from (1) to (2) is proven in proposition 1.3.3 combined with proposition 1.3.6, for the other, we refer to Ulbrich's theorem [90]. The last statement is proven in proposition 1.3.4. □

1.4 Hopf-Galois deformation of Hopf *-algebras

In this last section of the first chapter, we will extend the results to the case of Galois objects on Hopf *-algebras. Where the results for Hopf algebras are well known and clearly written in literature, for Hopf *-algebras, this is less the case. We try to give a clear description, the results in [23, 38, 39] were helpful.

Before we have a closer look to theory of Hopf *-algebras, we rephrase the second section of this chapter for Hopf algebras with a bijective antipode. As Hopf *-algebras satisfy the condition $S \circ * \circ S \circ * = \mathrm{id}$, the results of this first subsection are valid for Hopf *-algebras as well.

1.4.1 Galois objects on Hopf algebras with a bijective antipode.

In this subsection, H_1 is always a Hopf algebra with bijective antipode and B is a left H_1-Galois object with coaction $\beta_1 : B \to H_1 \odot B$. We will use the notation $H_2 = \widetilde{{}^B H_1}$. First we will prove that in this case, the inverse B' of B in the groupoid of bi-Galois objects is isomorphic with B^{op} and hence characterize H_1 and H_2 as $B \underset{H_2}{\boxdot} B^{op}$ resp. $B^{op} \underset{H_1}{\boxdot} B$.

Proposition 1.4.1 ([87]). *There exists a well defined right coaction*

$$\delta_1 : B^{op} \to B^{op} \odot H_1 : b^{op} \mapsto (b_{(0)})^{op} \otimes S^{-1}(b_{(-1)})$$

with $\beta_1(b) = b_{(-1)} \otimes b_{(0)}$ making B^{op} a right H_1-Galois object.

Proof. It is easy to see that

$$\begin{aligned} (\delta_1 \otimes \mathrm{id}_{H_1})\delta_1(b^{op}) &= (b_{(0)})^{op} \otimes S^{-1}(b_{(-1)}) \otimes S^{-1}(b_{(-2)}) \\ &= (b_{(0)})^{op} \otimes (S^{-1}(b_{(-1)}))_{(1)} \otimes (S^{-1}(b_{(-1)}))_{(2)} \\ &= (\mathrm{id}_B \otimes \Delta_1)\delta_1(b^{op}) \end{aligned}$$

and that

$$(\mathrm{id}_{B^{op}} \otimes \varepsilon)\delta_1(b^{op}) = (b_{(0)})^{op}\varepsilon(S^{-1}(b_{(-1)})) = (b_{(0)})^{op}\varepsilon(b_{(-1)}) = b^{op}.$$

Hence B^{op} is a right H-comodule. Moreover

$$\begin{aligned} \delta_1(b^{op}b'^{op}) &= \delta_1((b'b)^{op}) = (b'_{(0)}b_{(0)})^{op} \otimes S^{-1}(b'_{(-1)}b_{(-1)}) \\ &= b^{op}_{(0)}(b'_{(0)})^{op} \otimes S^{-1}(b_{(-1)})S^{-1}(b'_{(-1)}) \\ &= \delta_1(b^{op})\delta_1(b'^{op}) \end{aligned}$$

proving δ_1 to be multiplicative. Finally let $\phi : B^{op} \odot H_1 \to B^{op} \odot B^{op}$ be such that $\phi = (\mathrm{id}^{op} \otimes \mathrm{id}^{op})\sigma \circ T_\beta^{-1} \circ \sigma \circ (\mathrm{id}^{op} \otimes S)$ where $\mathrm{id}^{op}(b^{op}) = b$ and $\mathrm{id}^{op}(b) = b^{op}$.

Then it is a bijection as composition of bijections and moreover, we have

$$\begin{aligned}\phi \circ R_{\delta_1}(b^{op} \otimes b'^{op}) &= \phi((b'_{(0)}b)^{op} \otimes S^{-1}(b'_{(-1)})) \\ &= (\mathrm{id}^{op} \otimes \mathrm{id}^{op})\sigma \circ T_\beta^{-1}(b'_{(-1)} \otimes b'_{(0)}b) \\ &= b^{op} \otimes b'^{op}\end{aligned}$$

proving $\phi = R_{\delta_1}^{-1}$ and hence that B^{op} is a right H_1-Galois object.

□

Remark 1.4.2. *It is easy to see that* $\phi(1_B \otimes h) = (\mathrm{id}^{op} \otimes \mathrm{id}^{op})\sigma \circ \gamma \circ S(h)$*. We will denote this map* $\gamma' : H_1 \to B \odot B^{op} : h \mapsto \phi(1_B \otimes h)$*. One can check that then indeed* $R_{\delta_1}^{-1}(b^{op} \otimes h) = (b^{op} \otimes 1_H)\gamma'(h)$*.*

Proposition 1.4.3. *Let* $\beta_2 : B \to B \odot H_2$ *be the coaction making* B *a* (H_1-H_2)*-bi-Galois object. Let* $\delta_1 : B^{op} \to B^{op} \odot H_1$ *be the coaction of the previous proposition. Then*

$$\delta_2 : B^{op} \to H_2 \odot B^{op} : b^{op} \mapsto (\mathrm{id} \otimes \gamma')\delta_1(b^{op})$$

with $\gamma' : H_1 \to B \odot B^{op} : h \mapsto (\mathrm{id}^{op} \otimes \mathrm{id}^{op})\sigma \circ \gamma \circ S(h)$ *makes* B^{op} *a* (H_2-H_1)*-bi-Galois object.*

Proof. From the previous remark, one sees that this is implied by lemma 1.2.11 and theorem 1.2.8. □

Proposition 1.4.4. *We have the following isomorphisms of Hopf algebras:*

$$\mathrm{id}_{B^{op}} \otimes \mathrm{id}_B : H_2 \to B^{op} \underset{H_1}{\boxdot} B;$$

$$\gamma : H_1 \to B \underset{H_2}{\boxdot} B^{op}.$$

Proof. To prove the first isomorphism, note that both H_2 and $B^{op} \underset{H_1}{\boxdot} B$ are subalgebras of $B^{op} \odot B$. We will prove that for $b_i \otimes b'_i \in B^{op} \odot B$, the condition $(\mathrm{id} \otimes \beta_1)(b_i \otimes b'_i) = (\delta_1 \otimes \mathrm{id})(b_i \otimes b'_i)$ is equivalent with $\lambda(b_i \otimes b'_i) = 1 \otimes b_i \otimes b'_i$. Suppose first that $b_i \otimes b'_i$ satisfies the first condition. Then we have

$$b_i \otimes b'_{i(-1)} \otimes b'_{i(0)} = b_{i(0)} \otimes S^{-1}(b_{i(-1)}) \otimes b'_i$$

and hence

$$\begin{aligned}\lambda(b_i \otimes b_i') &= b_{i(-1)} b'_{i(-1)} \otimes b_{i(0)} \otimes b'_{i(0)} \\ &= b_{i(-1)} S^{-1}(b_{i(-2)}) \otimes b_{i(0)} \otimes b'_i \\ &= 1 \otimes b_i \otimes b'_i.\end{aligned}$$

If $b_i \otimes b_i'$ satisfies the second condition, we have

$$\begin{aligned}b_i \otimes b'_{i(-1)} \otimes b'_{i(0)} &= b_{i(0)} \otimes S^{-1}(b_{i(-1)}) b_{i(-2)} b'_{i(-1)} \otimes b'_{i(0)} \\ &= b_{i(0)} \otimes S^{-1}(b_{i(-1)}) \otimes b'_i\end{aligned}$$

proving the second implication. Finally, we know already that $\gamma(H_1) \subset B \odot B^{op}$ and in the proof of theorem 1.2.14 we have proven that $\gamma : H_1 \to (B \odot B)^{co\tilde{H}} \subset B \odot B^{op}$ is an isomorphism. Analogously as above, one can prove that $(B \odot B)^{co\tilde{H}} \cong B \underset{H_2}{\boxdot} B^{op}$. One can check easily that the isomorphism are compatible with the Hopf algebra structure. $\square$

WIth this proposition, we can prove that B^{op} is indeed the inverse of B in the groupoid of bi-Galois objects.

Proposition 1.4.5. *Let B' be the the inverse of B in the groupoid of bi-Galois objects. Then $B' \cong B^{op}$.*

Proof. One has the following isomorphisms:

$$B' = B' \underset{H_1}{\boxdot} H_1 \cong B' \underset{H_1}{\boxdot} B \underset{H_2}{\boxdot} B^{op} \cong H_2 \underset{H_2}{\boxdot} B^{op} \cong B^{op}$$

where the second isomorphism follows from proposition 1.4.4 and the rest from proposition 1.3.6. $\square$

Proposition 1.4.6. *If H_1 has a bijective antipode, so has H_2.*

Proof. For $x_i \otimes y_i \in H_2$, note that

$$
\begin{aligned}
&(\mathrm{id} \otimes \beta_1)\Big(y_{i(0)} \otimes l_j(S^{-1}(y_{i(-1)}))x_i r_j(S^{-1}(y_{i(-1)}))\Big)\\
&= y_{i(0)} \otimes \big(l_j(S^{-1}(y_{i(-1)}))x_i r_j(S^{-1}(y_{i(-1)}))\big)_{(-1)}\\
&\qquad\qquad \otimes \big(l_j(S^{-1}(y_{i(-1)}))x_i r_j(S^{-1}(y_{i(-1)}))\big)_{(0)}\\
&= y_{i(0)} \otimes S^{-1}(y_{i(-1)})x_{i(-1)}y_{i(-3)} \otimes l_j(S^{-1}(y_{i(-2)}))x_{i(0)} r_j(S^{-1}(y_{i(-2)}))\\
&= y_{i(0)} \otimes S^{-1}(y_{i(-1)})S(y_{i(-4)})y_{i(-3)} \otimes l_j(S^{-1}(y_{i(-2)}))x_i r_j(S^{-1}(y_{i(-2)}))\\
&= y_{i(0)} \otimes S^{-1}(y_{i(-1)}) \otimes l_j(S^{-1}(y_{i(-2)}))x_i r_j(S^{-1}(y_{i(-2)}))\\
&= (\delta_1 \otimes \mathrm{id})\Big(y_{i(0)} \otimes l_j(S^{-1}(y_{i(-1)}))x_i r_j(S^{-1}(y_{i(-1)}))\Big)
\end{aligned}
$$

where we used that $\delta_1(x_i) \otimes y_i = x_i \otimes \beta_1(y_i)$ in the third equality. Hence we can define a map $R : H_2 \to H_2 : x_i \otimes y_i \mapsto y_{i(0)} \otimes l_j(S^{-1}(y_{i(-1)}))x_i r_j(S^{-1}(y_{i(-1)}))$. We will prove that $R = S_2^{-1}$ where S_2 is the antipode of H_2. Note now that

$$
\begin{aligned}
&R \circ S_2(x_i \otimes y_i)\\
&= R\big(l_j(x_{i(-1)})y_i r_j(x_{i(-1)}) \otimes x_{i(0)}\big)\\
&= x_{i(0)} \otimes l_k(S^{-1}(x_{i(-1)}))l_j(x_{i(-2)})y_i r_j(x_{i(-2)}) r_k(S^{-1}(x_{i(-1)}))\\
&= x_{i(0)} \otimes l_k(S^{-1}(x_{i(-1)})x_{i(-2)})y_i r_k(S^{-1}(x_{i(-1)})x_{i(-2)})\\
&= x_i \otimes y_i
\end{aligned}
$$

and

$$
\begin{aligned}
&S_2 \circ R(x_i \otimes y_i)\\
&= S_2(y_{i(0)} \otimes l_j\big(S^{-1}(y_{i(-1)}))x_i r_j(S^{-1}(y_{i(-1)}))\big)\\
&= l_k(y_{i(-1)})l_j(S^{-1}(y_{i(-2)}))x_i r_j(S^{-1}(y_{i(-2)}))r_k(y_{i(-1)}) \otimes y_{i(0)}\\
&= l_k(y_{i(-1)}S^{-1}(y_{i(-2)}))x_i r_k(y_{i(-1)}S^{-1}(y_{i(-2)})) \otimes y_{i(0)}\\
&= x_i \otimes y_i.
\end{aligned}
$$

This proves the statement. □

We have just proved that the Hopf-Galois deformation preserves the property of having bijective antipode. Hence we can say that two Hopf algebras can only be Hopf-Galois equivalent if they have both bijective antipode, or both not bijective antipode.

1.4.2 Galois objects on Hopf *-algebras

In this subsection we proof that the Hopf-Galois deformation of a Hopf *-algebra is again a Hopf *-algebra. At the end of this subsection, we also prove that Hopf-Galois equivalent Hopf *-algebras have equivalent strict monoidal *-categories of *-comodules which is the *-algebraic version of theorem 1.3.11.

Definition 1.4.7. *Let H be a Hopf *-algebra, B a left H-Galois object which is a *-algebra and such that the coaction $\beta : B \to H \odot B$ is a *-morphism. Then B is called a left H-*-Galois object.*

In the rest of the section, H will be a Hopf *-algebra and B is a left H-*-Galois object with coaction $\beta_1 : B \to H \odot B : b \mapsto b_{\widetilde{(-1)}} \otimes b_{\widetilde{(0)}}$. To prove that $H_2 = \widetilde{^B H}$ is a Hopf *-algebra as well, we will consider $\widetilde{^B H}$ as $\widetilde{^B H} = B^{op} \underset{H}{\boxdot} B$ and put an appropriate *-structure on B^{op} such that $\delta_1 : B^{op} \to B^{op} \odot H$ is a *-morphism for this *-structure. Therefore we need some preliminary work about the Grunspan map θ [55, 86].

Proposition 1.4.8. *The Grunspan map $\theta : B \to B : b \mapsto l_j(S(b_{(-1)}))b_{(0)}r_j(S(b_{(-1)}))$ is an automorphism of B.*

In the classical case, this map is the identity. Indeed, if B is commutative, one has

$$l_j(S(b_{(-1)}))b_{(0)}r_j(S(b_{(-1)})) = b_{(0)}l_j(S(b_{(-1)}))r_j(S(b_{(-1)})) = b_{(0)}\varepsilon(S(b_{(-1)})) = b.$$

For proving this proposition, we need a lemma:

Lemma 1.4.9. *For $b \in B$, $l_j(S(b_{(-1)}))b_{(0)} \otimes r_j(S(b_{(-1)})) \in \mathbb{C} \odot B \subset B \odot B$.*

Proof. We have:

$$\begin{aligned}
&\beta_1\big(l_j(S(b_{(-1)}))b_{(0)}\big) \otimes r_j(S(b_{(-1)})) \\
&= l_j(S(b_{(-2)}))_{(-1)}b_{(-1)} \otimes l_j(S(b_{(-2)}))_{(0)}b_{(0)} \otimes r_j(S(b_{(-2)})) \\
&= S(b_{(-2)})b_{(-1)} \otimes l_j(S(b_{(-3)}))b_{(0)} \otimes r_j(S(b_{(-3)})) \\
&= 1 \otimes l_j(S(b_{(-1)}))b_{(0)} \otimes r_j(S(b_{(-1)}))
\end{aligned}$$

and as β_1 is ergodic, the statement is proven. □

With this result, we can prove proposition 1.4.8.

Proof of proposition 1.4.8. We have

$$\begin{aligned}
\theta(bb') &= l_j(S(b_{(-1)}b'_{(-1)}))b_{(0)}b'_{(0)}r_j(S(b_{(-1)}b'_{(-1)})) \\
&= l_j(S(b'_{(-1)}))l_k(S(b_{(-1)}))b_{(0)}b'_{(0)}r_k(S(b_{(-1)}))r_j(S(b'_{(-1)})) \\
&= l_j(S(b'_{(-1)}))b'_{(0)}l_k(S(b_{(-1)}))b_{(0)}r_k(S(b_{(-1)}))r_j(S(b'_{(-1)})) \\
&= l_k(S(b_{(-1)}))b_{(0)}r_k(S(b_{(-1)}))l_j(S(b'_{(-1)}))b'_{(0)}r_j(S(b'_{(-1)})) \\
&= \theta(b)\theta(b') \qquad (1.4.1)
\end{aligned}$$

where we used the previous lemma in the third and fourth equalities. To prove that θ is bijective, we define the following map:

$$\theta' : B \to B : b \to l_j(S^{-2}(b_{(-1)}))b_{(0)}r_j(S^{-2}(b_{(-1)})).$$

Note first that $l_j(S^{-2}(b_{(-1)})) \otimes b_{(0)}r_j(S^{-2}(b_{(-1)})) \in B \odot \mathbb{C}$. Indeed, one has

$$\begin{aligned}
&l_j(S^{-2}(b_{(-1)})) \otimes \beta_1\big(b_{(0)}r_j(S^{-2}(b_{(-1)}))\big) \\
&= l_j(S^{-2}(b_{(-2)})) \otimes b_{(-1)}r_j(S^{-2}(b_{(-2)}))_{(-1)} \otimes b_{(0)}r_j(S^{-2}(b_{(-1)}))_{(0)} \\
&= l_j(S^{-2}(b_{(-3)})) \otimes b_{(-1)}S(S^{-2}(b_{(-2)})) \otimes b_{(0)}r_j(S^{-2}(b_{(-3)})) \\
&= l_j(S^{-2}(b_{(-3)})) \otimes b_{(-1)}S^{-1}(b_{(-2)}) \otimes b_{(0)}r_j(S^{-2}(b_{(-3)})) \\
&= l_j(S^{-2}(b_{(-1)})) \otimes 1 \otimes b_{(0)}r_j(S^{-2}(b_{(-1)}))
\end{aligned}$$

proving the claim as β_1 is ergodic. Then we have

$$
\begin{aligned}
&\theta\circ\theta'(b)\\
&= \theta\Big(l_j(S^{-2}(b_{(-1)}))b_{(0)}r_j(S^{-2}(b_{(-1)}))\Big)\\
&= \theta\Big(l_j(S^{-2}(b_{(-1)}))\Big)b_{(0)}r_j(S^{-2}(b_{(-1)}))\\
&= l_k\Big(S\big(l_j(S^{-2}(b_{(-1)}))_{(-1)}\big)\Big)l_j(S^{-2}(b_{(-1)}))_{(0)}r_k\Big(S\big(l_j(S^{-2}(b_{(-1)}))_{(-1)}\big)\Big)\\
&\qquad b_{(0)}r_j(S^{-2}(b_{(-1)}))\\
&= l_k\big(S(S^{-2}(b_{(-2)}))\big)l_j(S^{-2}(b_{(-1)}))r_k\big(S(S^{-2}(b_{(-2)}))\big)\\
&\qquad b_{(0)}r_j(S^{-2}(b_{(-1)}))\\
&= l_k(S^{-1}(b_{(-2)}))l_j(S^{-2}(b_{(-1)}))b_{(0)}r_j(S^{-2}(b_{(-1)}))r_k(S^{-1}(b_{(-2)}))\\
&= l_k\big(S^{-2}(S(b_{(-2)})b_{(-1)})\big)b_{(0)}r_k\big(S^{-2}(S(b_{(-2)})b_{(-1)})\big)\\
&= b
\end{aligned}
$$

where we used that $l_j(S^{-2}(b_{(-1)}))\otimes b_{(0)}r_j(S^{-2}(b_{(-1)}))\in B\odot\mathbb{C}$ in the fifth equality. Analogously $\theta'\circ\theta(b)=b$. Hence $\theta'=\theta^{-1}$ and together with (1.4.1) this proves that θ is an automorphism. $\square$

Lemma 1.4.10. *With $\gamma: H\to B\odot B^{op}$ the morphism such that $T_{\beta_1}(\gamma(h))=h\otimes 1_B$, one has $\gamma(h)^{*\otimes *}=\sigma(\gamma(S^{-1}(h^*)))$.*

Proof. One has

$$
\begin{aligned}
T_{\beta_1}\big(r_i(h^*)^*\otimes l_i(h^*)^*\big) &= r_i(h^*)^*_{(-1)}\otimes r_i(h^*)^*_{(0)}l_i(h^*)^*\\
&= S(h^*_{(2)})^*\otimes r_i(h^*_{(1)})^*l_i(h^*_{(1)})^*\\
&= S^{-1}(h_{(2)})\otimes (l_i(h^*_{(1)})r_i(h^*_{(1)}))^*\\
&= S^{-1}(h)\otimes 1
\end{aligned}
$$

proving that $T_{\beta_1}(\sigma(\gamma(h^*)^{*\bigotimes *})) = T_{\beta_1}(\gamma(S^{-1}(h)))$. Hence $\sigma(\gamma(h^*)^{*\bigotimes *}) = \gamma(S^{-1}(h))$ which implies the statement. $\square$

Proposition 1.4.11. *B^{op} is a *-algebra with involution $(b^{op})^\star = \theta(b^*)^{op}$.*

Proof. We prove that $\star$ is an involution. We have

$$\begin{aligned}
((b^{op})^\star)^\star &= (\theta(b^*)^{op})^\star \\
&= \left(l_j(S(b^*_{(-1)}))b^*_{(0)}r_j(S(b^*_{(-1)}))\right)^* \\
&= r_j(S(b^*_{(-1)}))^* b_{(0)} l_j(S(b^*_{(-1)}))^* \\
&= r_j(S^{-1}(b_{(-1)})^*)^* b_{(0)} l_j(S^{-1}(b_{(-1)})^*)^* \\
&= l_j(S^{-2}(b_{(-1)}))b_{(0)} r_j(S^{-2}(b_{(-1)})) \\
&= \theta^{-1}(b).
\end{aligned}$$

Hence $((b^{op})^\star)^\star = b^{op}$. Moreover, as θ is an automorphism of B, one has

$$\begin{aligned}
\left(b^{op}b'^{op}\right)^\star &= \left((b'b)^{op}\right)^\star \\
&= \theta\left((b'b)^*\right)^{op} \\
&= \left(\theta(b^*)\theta(b'^*)\right)^{op} \\
&= \theta(b'^*)^{op}\theta(b^*)^{op} \\
&= (b'^{op})^\star(b^{op})^\star
\end{aligned}$$

proving that $\star$ is a well defined involution. $\square$

Now we have defined a new *-structure on B^{op}, we are ready to define the *-structure on H_2.

Proposition 1.4.12. *We have*

1. *δ_1 is a *-morphism,*
2. *H_2 is a *-algebra induced by the involution on $B^{op} \odot B$,*

3. *$\gamma : H_1 \to B \odot B^{op}$ is a *-morphism for the respective involutions extending the isomorphism of proposition 1.4.4 to a *-isomorphism,*

4. *β_2 and δ_2 are *-morphisms.*

Proof. 1. We have

$$\begin{aligned}
&\delta_1((b^{op})^\star)\\
&= l_j(S(b^*_{(-1)}))b^*_{(0)}\delta_1\big(r_j(S(b^*_{(-1)}))\big)\\
&= l_j(S(b^*_{(-1)}))b^*_{(0)}r_j(S(b^*_{(-1)}))_{(0)} \otimes S^{-1}(r_j(S(b^*_{(-1)}))_{(-1)})\\
&= l_j(S(b^*_{(-1)}))b^*_{(0)}r_j(S(b^*_{(-1)})) \otimes S(b^*_{(-2)})\\
&= \theta(b^*_{(0)}) \otimes S^{-1}(b_{(-1)})^*\\
&= \delta_1(b^{op})^{\star\otimes *}
\end{aligned}$$

proving the claim.

2. As both β_1 and δ_1 are *-morphisms, it is easy to see that proposition 1.4.4 implies that H_2 is a well defined Hopf *-algebra.

3. From lemma 1.4.10, we have

$$\begin{aligned}
&\gamma(h)^{*\otimes\star}\\
&= l_i(h)^* \otimes \theta(r_i(h)^*)\\
&= r_i(S^{-1}(h^*)) \otimes \theta(l_i(S^{-1}(h^*)))\\
&= r_i(S^{-1}(h^*)) \otimes l_j\big(S(l_i(S^{-1}(h^*))_{(-1)})\big)\\
&\qquad\qquad l_i(S^{-1}(h^*))_{(0)}r_j\big(S(l_i(S^{-1}(h^*))_{(-1)})\big)\\
&= r_i(S^{-1}(h^*_{(1)})) \otimes l_j(h^*_{(2)})l_i(S^{-1}(h^*_{(1)}))r_j(h^*_{(2)})\\
&= l_j(h^*_{(2)})l_i(S^{-1}(h^*_{(1)}))r_i(S^{-1}(h^*_{(1)})) \otimes r_j(h^*_{(2)})\\
&= l_j(h^*) \otimes r_j(h^*)\\
&= \gamma(h^*)
\end{aligned}$$

where we used lemma 1.4.9 in the fifth equality.

4. As $\beta_2 = (\gamma \otimes \mathrm{id})\beta_1$ is the composition of *-morphisms, it is a *-morphism as well. δ_2 is constructed from β_2 in the same way δ_1 is constructed from β_1. Hence copying mutatis mutandis the proof of the first result of this proposition, we're done.

□

We now proved that the Hopf-Galois deformation of a Hopf *-algebra is again a Hopf *-algebra. In subsection 1.3.1, we proved that Hopf-Galois equivalent Hopf algebras have equivalent strict monoidal categories of comodules. We can upgrade this result to Hopf *-algebras.

Definition 1.4.13 ([23]).
- *Let H be a Hopf *-algebra and V a unital right H-comodule with coaction $\alpha_V : V \to V \odot H$. Denoting by $\overline{V}$ the conjugate vector space of V and $j_V : V \to \overline{V}$ the semilinear isomorphism, $\overline{V}$ is a H-comodule with coaction $\alpha_{\overline{V}} = (j_V \otimes *)\alpha_V \circ j_V^{-1}$. If V is a Hilbert space such that the scalar product $s : \overline{V} \odot V \to \mathbb{C}$ is a morphism of right H-comodules, V is called a unitary H-comodule or H-*-comodule.*
- *A Haar measure for H is a linear map $h : V \to \mathbb{C}$ such that $(\mathrm{id} \otimes h)\Delta(a) = h(a)1_H = (h \otimes \mathrm{id})\Delta(a)$ for every $a \in H$.*
- *A Haar measure for a left H-*-Galois object is a linear map ω such that $(\mathrm{id} \otimes \omega)\beta(b) = \omega(b)1_H$ for every $b \in B$. It is called positive if $\omega(b^*b) \geq 0$ for every $b \in B$ and called faithful if $\omega(b^*b) > 0$ for every $b \in B \setminus \{0\}$.*

Theorem 1.4.14. *Let H be a Hopf *-algebra with Haar measure h and B a left H-*-Galois object with positive faithful Haar measure ω. Then*

1. *([23]) if V is a finite dimensional unital right H-*-comodule, then $\tilde{V}$ is a finite dimensional unital right $\tilde{H}$-*-comodule. ;*
2. *if A is a unital right H-*-comodule-algebra and α a *-coaction, then $\tilde{A} = A \underset{H}{\boxdot} B$ is a right $\tilde{H}$-*-comodule-algebra.*

Proof. 1. Denote by $s : \overline{V} \odot V \to \mathbb{C}$ the bilinear map on V. Then define the following map:

$$\mu_V : \overline{V \underset{H}{\boxdot} B} \to \overline{V} \underset{H}{\boxdot} B : \overline{v_i \otimes z_i} \mapsto \overline{v}_i \otimes z_i^*.$$

It is well defined as

$$\begin{aligned}
(\mathrm{id}\otimes\beta)\mu_V(\overline{v_i\otimes z_i}) &= \overline{v}_i\otimes\beta(z_i^*)\\
&= (j_V\otimes *\otimes *)(v_i\otimes\beta(z_i))\\
&= (j_V\otimes *\otimes *)(\alpha_V(v_i)\otimes z_i)\\
&= (j_V\otimes *\otimes \mathrm{id})\alpha_V\circ j_V^{-1}(\overline{v}_i)\otimes z_i^*\\
&= (\alpha_{\overline{V}}\otimes\mathrm{id})\mu_V(\overline{v_i\otimes z_i}).
\end{aligned}$$

Moreover, μ_V is an isomorphism, which is easily verified.
Next, using proposition 1.3.3, we can establish the following bilinear map:

$$\tilde{s}:\overline{V\underset{H}{\boxdot}B}\odot V\underset{H}{\boxdot}B\overset{\mu_V\otimes\mathrm{id}_B}{\cong}\overline{V}\underset{H}{\boxdot}B\odot V\underset{H}{\boxdot}B\overset{\varphi_2}{\cong}(\overline{V}\odot V)\underset{H}{\boxdot}B\xrightarrow{s\otimes\mathrm{id}_B}\mathbb{C}\underset{H}{\boxdot}B\overset{\varphi_1}{\cong}\mathbb{C}$$

$$\overline{v_i\otimes z_i}\otimes v_j\otimes z'_j\mapsto\omega(z_i^*z'_i)$$

where v_i is an orthonormal basis of V with respect to s and $\sum_{i,j}s(v_i,v_j)z_i^*z'_j=\sum_i z_i^*z'_i=\omega(z_i^*z'_i)1_B$. Moreover, as ω and s are positive and faithful, so is $\tilde{s}$ by construction. Furthermore denoting with λ the codiagonal action of $\tilde{H}$ on $\overline{V\underset{H}{\boxdot}B}\odot V\underset{H}{\boxdot}B$ and β the coaction of $\tilde{H}$ on B, one has

$$\begin{aligned}
(\tilde{s}\otimes\mathrm{id})\lambda(\overline{v_i\otimes z_i}\otimes v_j\otimes z'_j) &= \tilde{s}\big(\overline{v_i\otimes z_{i(0)}}\otimes v_j\otimes z'_{j(0)}\big)\otimes z^*_{i(1)}z'_{j(1)}\\
&= \sum_i z^*_{i(0)}z'_{i(0)}\odot z^*_{i(1)}z'_{i(1)}\\
&= \beta(\sum_i z_i^*z'_i)\\
&= \beta\Big(\tilde{s}\big(\overline{v_i\otimes z_i}\otimes v_j\otimes z'_j\big)\Big)
\end{aligned}$$

where again $(v_i)_i$ is an orthonormal basis of V.

2. As α and $\beta_1: B\to H\odot B$ are *-coactions, $A\underset{H}{\boxdot}B$ is a *-algebra and the *-coaction $\beta_2: B\to B\odot\tilde{H}$ gives $A\underset{H}{\boxdot}B$ the structure of a $\tilde{H}$-*-comodule-algebra.

□

This theorem shows that the equivalence of strict monoidal categories of comodules obtained in section 1.3 can be lifted to an equivalence of strict monoidal *-categories of *-comodules in the case of Hopf *-algebras (for the exact definitions of *-category, monoidal *-category and *-functors, we refer to [23]).

1.5 Conclusion

In this first chapter, we described the deformation procedure of Hopf algebras (developed in [87]) and of Hopf *-algebras (developed in [23]). We reminded that this deformation induces an equivalence relation on the set of Hopf algebras. Moreover two Hopf (*)-algebras which are equivalent have equivalent strict monoidal (*-)categories of (*-)comodules. It is this last approach of using the equivalence of categories which will be followed in expanding this theory to the Compact Quantum Group world. This will be done in the next chapter.

Chapter 2

Compact quantum groups and Monoidal equivalences

In the first chapter we followed the algebraic path to describe Hopf-Galois deformation. In this second chapter, we give an alternative description following an analytical way. It is this approach that will be used to define a deformation of spectral triples in the third chapter.

This chapter is organized as follows. In the first section, we recapitulate some standard concepts and results in the theory of C^*-algebras. In the second we recall the theory of compact quantum groups (CQG) and emphasize in the third on action of full quantum multiplicity, which turns out to be the analytical counterpart of coactions of Galois objects on which we focused in the first chapter. Finally we recall the concepts of monoidal equivalence in the fourth and last section, the analytical counterpart of Hopf-Galois deformation.

2.1 Preliminaries on C^*-algebras

Before we start, we will clarify some notations and basic notions. We refer to the books [77] and [45] for further explanantion. Given a Hilbert space $\mathcal{H}$, the inner product $\langle\cdot,\cdot\rangle$ is linear in the second variable. Moreover,

- a (linear) functional on $\mathcal{H}$ is a continuous linear map $f : \mathcal{H} \to \mathbb{C}$;

- the norm of a functional f on $\mathcal{H}$ is $\|f\| = \sup\{|f(\xi)| \mid \xi \in \mathcal{H}, \|\xi\| = 1\}$;

Given $\xi, \eta \in \mathcal{H}$ we define

- ξ^* to be the functional $\mathcal{H} \to \mathbb{C} : \eta \mapsto \langle \xi, \eta \rangle$,
- $\xi\eta^*$ to be the rank one operator $\mathcal{H} \to \mathcal{H} : \zeta \mapsto \langle \eta, \zeta \rangle \xi$.

For two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ with inner products $\langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2$ the tensor product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is the completion of

$$\mathcal{H}_1 \odot \mathcal{H}_2 = \{\sum_{i=1}^{n} \xi_i \otimes \eta_i \mid n \in \mathbb{N}, \xi_i \in \mathcal{H}_1, \eta_i \in \mathcal{H}_2\}$$

with inner product $\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle = \langle \xi_1, \eta_1 \rangle_1 \langle \xi_2, \eta_2 \rangle_2$.

Furthermore, we will denote by $B(\mathcal{H})$ resp. $\mathcal{K}(\mathcal{H})$ the bounded resp. compact operators on $\mathcal{H}$. For a bounded or unbounded operator D on $\mathcal{H}$, $\sigma(D)$ will be used to denote its spectrum.
A functional on a C^*-algebra A is a continuous linear map $f : A \to \mathbb{C}$. The norm of a functional f on A is $\|f\| = \sup\{|f(a)| \mid a \in A, \|a\| = 1\}$ and it is called positive if $f(a^*a) \geq 0$ for all $a \in A$. A state on $\mathcal{H}$ is a positive functional of norm 1.
We use $\omega_{\xi,\eta}$ to denote the functional $B(\mathcal{H}) \to \mathbb{C} : a \mapsto \langle \xi, a\eta \rangle$ where $\xi, \eta \in \mathcal{H}$.
Moreover, for a subset B of a C^*-algebra A, we define

- $\langle B \rangle$ to be the linear span of B,
- $[B]$ the closed linear span of B,
- $S(B)$ the *-algebra generated by elements of B,
- $C^*(B)$ the C^*-subalgebra of A generated by elements of B.

On C^*-algebras, one can define different tensor products. Algebraically, if A and B are two C^*-algebras, $A \odot B = \{\sum_{i=1}^{n} a_i \otimes b_i \mid n \in \mathbb{N}, a_i \in A, b_i \in B\}$ is the algebraic tensor product with product $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ and involution $(a \otimes b)^* = a^* \otimes b^*$. However, one can put different C^*-norms on $A \odot B$. Two of them are mostly used.

Definition 2.1.1. *Let A, B be two C^* algebras.*

- *The minimal C^*-norm on $A \odot B$ is defined by*

$$\|\sum_i a_i \otimes b_i\|_{\min} = \sup_{\pi_A, \pi_B} \{\|\sum_i \pi_A(a_i) \otimes \pi_B(b_i)\|_{B(\mathcal{H}_1 \otimes \mathcal{H}_2)}\}$$

  where π_A resp. π_B are representations of A resp. B on Hilbert spaces $\mathcal{H}_1$ resp. $\mathcal{H}_2$.

- *The maximal C^*-norm on $A \odot B$ is defined by*

$$\|\sum a_i \otimes b_i\|_{\max} = \sup_{\pi} \{\|\sum_i \pi(a_i \otimes b_i)\|_{B(\mathcal{H}_1 \otimes \mathcal{H}_2)}\}$$

  where π is a representation of $A \odot B$ on a Hilbert space $\mathcal{H}$.

The completion of $A \odot B$ in these respective norms are called minimal (or spatial) resp. maximal tensor product and denoted by $A \otimes_{\min} B$ resp. $A \otimes_{\max} B$.

One can prove that for an arbitrary C^*-norm $\|\cdot\|$ on $A \odot B$, the inequalities $\|\cdot\|_{\min} \leq \|\cdot\| \leq \|\cdot\|_{\max}$ hold. In the rest of this thesis, we will use the minimal C^*-algebraic tensor product and denote it for notational convenience by $\otimes$.

Having von Neumann algebras $M \subset B(\mathcal{H})$ and $N \subset B(\mathcal{K})$, we define the normal spatial tensor product to be the weak* closure of $M \odot N$ inside $B(\mathcal{H} \otimes \mathcal{K})$. This tensor product of von Neumann algebras is denoted by $\overline{\otimes}$.

We will also use the legnumbering notation in three and multiple tensor products: for $a \in A \otimes A$, we let $a_{12} = a \otimes 1_A$, $a_{23} = 1_A \otimes a_{23}$ and $a_{13} = (\mathrm{id} \otimes \tau)(a \otimes 1_A)$, all three elements in $A \otimes A \otimes A$ where $\tau(a \otimes b) = b \otimes a$.

Finally, we need the notion of Hilbert C^*-module. It was introduced by Kaplansky in [59] and further developed by Rieffel [81] and Paschke [76]. A very useful reference is [66].

Definition 2.1.2. *A pre-Hilbert C^*-module over a C^*-algebra A is a complex vector space E which is a left A-module equipped with an A-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to A$ satisfying:*

- $\langle x, x \rangle \geq 0$ *and* $\langle x, x \rangle = 0$ *if and only if* $x = 0$,
- $\langle y, x \rangle = \langle x, y \rangle^*$

- $\langle x, y + \lambda z\rangle = \langle x, y\rangle + \lambda\langle x, z\rangle$
- $\langle x, ay\rangle = a\langle x, y\rangle$ *for* $a \in A$.

A pre-Hilbert C^*-module over A is called a Hilbert C^*-module over A if it is complete with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$.

The most easy examples are the Hilbert spaces (where $E = \mathcal{H}$ and $A = \mathbb{C}$) and C^*-algebras (where $E = A$ and $\langle x, y\rangle = x^*y$). Other examples are the spaces $\mathcal{H} \otimes A$.

Proposition 2.1.3. *Let A be a C^*-algebra and $\mathcal{H}$ a Hilbert space. Then $\mathcal{H} \odot A = \langle \xi \otimes a \mid \xi \in \mathcal{H}, a \in A\rangle$ with $\langle \xi \otimes a, \eta \otimes b\rangle = \langle \xi, \eta\rangle a^*b$ is a pre-Hilbert C^*-module over A. The completion is denoted by $\mathcal{H} \otimes A$.*

We will use this Hilbert C^*-module $\mathcal{H} \otimes A$ in proposition 2.2.4 to give another description of a representation of a compact quantum group. The following definition describes the maps between Hilbert C^*-modules.

Definition 2.1.4. *Let A be a C^*-algebra and E a Hilbert C^*-modules over A. We call a map $T : E \to E$ adjointable if there exists a map $T^* : E \to E$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all x and y in E. The set of adjointable maps on E will be denoted by $B(E)$.*

We also use the multiplier algebra:

Definition 2.1.5 ([29])**.** *Let A be a (not necessarily unital) C^*-algebra. Considering A as a Hilbert C^*-module over itself, we define the multiplier algebra $\mathcal{M}(A)$ of A as $B(A)$.*

The multiplier algebra can also be realized as the set of two-sided multipliers in the enveloping von Neumann algebra of A. If B is the set of multipliers of A, $ab \in A$ and $ba \in A$ for all $a \in A, b \in B$.

2.2 Compact quantum groups and representations

In this section we give a short overview of the theory of compact quantum groups. The theory is essentially developed in [102], [105] and also explained in [70].

Definition 2.2.1 ([105]). *A compact quantum group* $\mathbb{G}$ *is a pair* $(C(\mathbb{G}), \Delta)$ *where* $C(\mathbb{G})$ *is a unital, seperable* C^**-algebra and* $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ *a unital* **-morphism such that*

1. $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$
2. $[\Delta(C(\mathbb{G}))(C(\mathbb{G}) \otimes 1)] = C(\mathbb{G}) \otimes C(\mathbb{G}) = [\Delta(C(\mathbb{G}))(1 \otimes C(\mathbb{G}))]$

implementing coassociativity and the cancellation properties.

Note that the suggestive notation $C(\mathbb{G})$ emphasizes the classical intuition where $C(\mathbb{G})$ is the algebra of continuous functions on a compact group. Using the Gelfand-Naimark theorem, every compact quantum group with commutative algebra is of this form, i.e. there exists a compact group G such that $C(\mathbb{G}) = C(G)$. If $C(\mathbb{G})$ is not commutative, there does not exist an underlying space.

Theorem 2.2.2 ([70, 102, 105]). *Let* $\mathbb{G}$ *be a compact quantum group. Then there exists a unique state* h *on* $C(\mathbb{G})$ *which is left and right invariant, i.e.*

$$(\mathrm{id} \otimes h)\Delta(x) = h(x)1_{C(\mathbb{G})} = (h \otimes \mathrm{id})\Delta(x)$$

for all $x \in C(\mathbb{G})$*. This state is called the Haar state of* $\mathbb{G}$*.*

In the classical case that $C(\mathbb{G}) = C(G)$ for a classical compact group G, the Haar state is the state on $C(G)$ obtained by integrating along the Haar measure i.e. the unique left and right invariant measure on G.

It is well known that, like compact groups, compact quantum groups have a rich representation theory ([70, 102, 105]). As we will see, that theory will play a crucial role.

Definition 2.2.3. *A right unitary representation of a compact quantum group* $\mathbb{G} = (C(\mathbb{G}), \Delta)$ *on a Hilbert space* $\mathcal{H}$ *is a unitary element* U *of* $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes C(\mathbb{G}))$ *satisfying*

$$(\mathrm{id} \otimes \Delta)U = U_{12}U_{13}.$$

Analogously, a left unitary representation of a compact quantum group $\mathbb{G} = (C(\mathbb{G}), \Delta)$ *on a Hilbert space* $\mathcal{H}$ *is a unitary element* U *of* $\mathcal{M}(C(\mathbb{G}) \otimes \mathcal{K}(\mathcal{H}))$ *satisfying*

$$(\Delta \otimes \mathrm{id})U = U_{13}U_{23}.$$

The dimension of $\mathcal{H}$ *is called the dimension of the representation.*

For now on, we will always work with right representation if not specified. Of course, all (general) results are also valid for left representations. Identifying $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes C(\mathbb{G}))$ with $\mathcal{B}(\mathcal{H} \otimes C(\mathbb{G}))$, the C^*-algebra of $C(\mathbb{G})$-linear adjointable maps on the Hilbert-C^*-module $\mathcal{H} \otimes C(\mathbb{G})$, we will see representations also from another perspective. Indeed, as a unitary representation under this isomorphism is determined by its restriction to $\mathcal{H} \otimes 1_{C(\mathbb{G})}$, we get the following proposition.

Proposition 2.2.4. *Let U be a unitary representation of a compact quantum group $\mathbb{G}$. Then the map*

$$u : \mathcal{H} \to \mathcal{H} \otimes C(\mathbb{G}) : \xi \to U(\xi \otimes 1_{C(\mathbb{G})})$$

satisfies

1. $\langle u(\xi), u(\eta) \rangle_{C(\mathbb{G})} = \langle \xi, \eta \rangle 1_{C(\mathbb{G})}$,
2. $(u \otimes \mathrm{id})u = (\mathrm{id} \otimes \Delta)u$,
3. $[\{u(\xi)(1 \otimes a) \mid \xi \in \mathcal{H}, a \in C(\mathbb{G})\}] = \mathcal{H} \otimes C(\mathbb{G})$,

for arbitrary $\xi, \eta \in \mathcal{H}$. Moreover, a map satisfying those three conditions induces a unique unitary representation on $\mathbb{G}$.

Proof. We first prove that a map u induced by a given irreducible representation U indeed satisfies the three conditions. As U is unitary in $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes C(\mathbb{G})) \cong \mathcal{B}(\mathcal{H} \otimes C(\mathbb{G}))$, we have

$$\begin{aligned} \langle u(\xi), u(\eta) \rangle_{C(\mathbb{G})} &= \langle U(\xi \otimes 1_{C(\mathbb{G})}), U(\eta \otimes 1_{C(\mathbb{G})}) \rangle_{C(\mathbb{G})} \\ &= \langle \xi \otimes 1_{C(\mathbb{G})}, \eta \otimes 1_{C(\mathbb{G})} \rangle_{C(\mathbb{G})} \\ &= \langle \xi, \eta \rangle 1_{C(\mathbb{G})} \end{aligned}$$

for arbitrary $\xi, \eta \in \mathcal{H}$. Moreover, for $\xi \in \mathcal{H}$,

$$(u \otimes \mathrm{id})u(\xi) = U_{12}U_{13}(\xi \otimes 1_{C(\mathbb{G})} \otimes 1_{C(\mathbb{G})})$$

$$= \big((\mathrm{id} \otimes \Delta)U\big)(\xi \otimes 1_{C(\mathbb{G})} \otimes 1_{C(\mathbb{G})}) = (\mathrm{id} \otimes \Delta)u(\xi)$$

and hence the second property is proven. Finally, note that, as U is unitary, the adjointable map on the Hilbert C^*-module $\mathcal{H} \otimes C(\mathbb{G})$

$$u' : \mathcal{H} \otimes C(\mathbb{G}) \to \mathcal{H} \otimes C(\mathbb{G}) : \sum_i \xi_i \otimes a_i \to \sum_i U(\xi_i \otimes a_i)$$

is surjective. Hence, taking an arbitrary element a in $\mathcal{H}\otimes C(\mathbb{G})$, we have $a = u'(a')$, for some $a' \in \mathcal{H} \otimes C(\mathbb{G})$. Taking a sequence $(\sum_{i=1}^{n_k} \xi_i^k \otimes a_i^k)_{k\in\mathbb{N}}$ in $\mathcal{H} \odot C(\mathbb{G})$ converging to a' and using the continuity of u', we have

$$\sum_{i}^{n_k} u(\xi_i^k)(1 \otimes a_i) = u'(\sum_{k=i}^{n_k} \xi_i^k \otimes a_i^k) \to u'(a') = a$$

proving the third statement.
Contrary, if we have a map $u : \mathcal{H} \to \mathcal{H} \otimes C(\mathbb{G})$ satisfying the conditions, we can extend it $C(\mathbb{G})$-linearly to a map

$$U' : \mathcal{H} \otimes C(\mathbb{G}) \to \mathcal{H} \otimes C(\mathbb{G}) : \sum_i \xi_i \otimes a_i \mapsto \sum_i u(\xi_i)(1 \otimes a_i)$$

using the continuity of u implied by condition 1. Note that $U' \in \mathcal{B}(\mathcal{H} \otimes C(\mathbb{G}))$. Using condition 1 resp. 3, we know that U' is isometric resp. surjective and hence unitary. Finally using the isomorphism $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes C(\mathbb{G})) \cong \mathcal{B}(\mathcal{H} \otimes C(\mathbb{G}))$, we find an element $U \in \mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes C(\mathbb{G}))$ and condition 2 implies the property $(\mathrm{id} \otimes \Delta)U = U_{12}U_{13}$. This ends the proof. □

Of course, we can make an analogous statement for left representations.

We also have a notion of tensor product of representations:

Definition 2.2.5. *Let U and V be representations of a compact quantum group $\mathbb{G} = (C(\mathbb{G}), \Delta)$ on Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ respectively. The tensor product $U \otimes V$ of U and V is defined as*

$$U \otimes V = U_{13}V_{23} \in \mathcal{M}(\mathcal{K}(\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes C(\mathbb{G})).$$

Definition 2.2.6. *For a compact quantum group $(C(\mathbb{G}), \Delta)$ and unitary representations U_1 and U_2 on Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ respectively, we consider the set*

$$\mathrm{Mor}(U^1, U^2) := \{S \in B(\mathcal{H}_2, \mathcal{H}_1) | (S \otimes 1_{C(\mathbb{G})})U^2 = U^1(S \otimes 1_{C(\mathbb{G})})\}$$

as the set of intertwiners between U_1 and U_2.

Definition 2.2.7. *A unitary representation $U \in \mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes C(\mathbb{G}))$ is called irreducible if* $\mathrm{Mor}(U, U) = \mathbb{C}1_{\mathcal{H}}$. *A unitary representation $V \in \mathcal{M}(\mathcal{K}(\mathcal{H}') \otimes C(\mathbb{G}))$ is unitary equivalent to U if there is a unitary operator in* $\mathrm{Mor}(U, V)$.

We have the following important result:

Theorem 2.2.8 ([70, 102, 105])**.** *Every irreducible representation of a compact quantum group is finite dimensional. Every unitary representation is unitarily equivalent to a direct sum of irreducible representations.*

Remark 2.2.9. *For a compact quantum group* $\mathbb{G}$*, we denote by* $\mathrm{Irred}(\mathbb{G})$ *the set of equivalence classes of irreducible representations of* $\mathbb{G}$ *and for* $x \in \mathrm{Irred}(\mathbb{G})$*, we will always take a unitary representative* $U^x \in B(\mathcal{H}_x) \otimes C(\mathbb{G})$*. By* ε*, we will denote the class of the trivial representation* $1_{C(\mathbb{G})}$*.*

In the following, we introduce contragredient representations. Let $\mathbb{G}$ be a compact quantum group. Woronowicz showed in [105] that in $\mathrm{Irred}(\mathbb{G})$, there exists an involution mapping a $x \in \mathrm{Irred}(\mathbb{G})$ to a representation $\overline{x}$ such that

$$\mathrm{Mor}(x \otimes \overline{x}, \varepsilon) \neq 0 \neq \mathrm{Mor}(\overline{x} \otimes x, \varepsilon)$$

and moreover that $U^{\overline{x}}$ is unique up to unitary equivalence. Taking $t_x \in \mathrm{Mor}(x \otimes \overline{x}, \varepsilon)$, $t_x \neq 0$ and identifying it with the element in $\mathcal{H}_x \otimes \mathcal{H}_{\overline{x}}$, we can define the antilinear map

$$j_x : \mathcal{H}_x \to \mathcal{H}_{\overline{x}} : \xi \mapsto (\xi^* \otimes 1)t_x. \tag{2.2.1}$$

Furthermore, let $Q_x = j_x^* j_x$, and normalize t_x so that $\mathrm{Tr}(Q_x) = \mathrm{Tr}(Q_x^{-1})$. Then Q_x is uniquely determined and t_x is determined up to a phase (i.e. a scalar of modulus one).

Definition 2.2.10 ([27])**.** *Let* $\mathbb{G}$ *be a compact quantum group and* $x \in \mathrm{Irred}(\mathbb{G})$*. The quantum dimension of* x *is defined to be* $\mathrm{Tr}(Q_x)$ *and denoted by* $\dim_q(x)$*.*

Note that $\dim_q(x) = t_x^* t_x$ and that $\dim_q(\overline{x}) = \dim_q(x) \geq \dim(x)$ and equality holds if and only if $Q_x = 1$. Moreover, there are some orthogonality relations for the irreducible representations of $\mathbb{G}$: for $\xi \in \mathcal{H}_x, \eta \in \mathcal{H}_y$, we have from [105]

$$(\mathrm{id} \otimes h)\Big((U^x(\xi \otimes 1_{C(\mathbb{G})}))\big(U^y(\eta \otimes 1_{C(\mathbb{G})})\big)^*\Big) = \frac{\delta_{x,y}\,\mathrm{id}_{\mathcal{H}_x}}{\dim_q(x)}\langle \eta, Q_x \xi\rangle$$

and

$$(\mathrm{id} \otimes h)\Big((U^x)^*(\xi \otimes 1_{C(\mathbb{G})})\big((U^y)^*(\eta \otimes 1)\big)\Big) = \frac{\delta_{x,y}\,\mathrm{id}_{\mathcal{H}_x}}{\dim_q(x)}\langle \eta, Q_x^{-1} \xi\rangle.$$

Remark 2.2.11. *For a compact quantum group* $\mathbb{G} = (C(\mathbb{G}), \Delta)$*, we will denote by* $\mathcal{O}(\mathbb{G})$ *the set of matrixcoefficients of all finite dimensional representations of* $\mathbb{G}$*. We have:*

$$\mathcal{O}(\mathbb{G}) = \langle (\omega_{\xi,\eta} \otimes \mathrm{id}_{C(\mathbb{G})})U^x | x \in \mathrm{Irred}(\mathbb{G}), \xi, \eta \in \mathcal{H}_x \rangle.$$

The following very non-trivial result is fundamental:

Theorem 2.2.12 ([102, 105])**.** $\mathcal{O}(\mathbb{G})$ *is a unital dense* **-subalgebra of* $C(\mathbb{G})$ *and restricting* Δ *to* $\mathcal{O}(\mathbb{G})$*,* $\mathcal{O}(\mathbb{G})$ *is a Hopf* **-algebra. Also, for a* $x \in \mathrm{Irred}(\mathbb{G})$*, let*

$\mathcal{O}(\mathbb{G})_x = \langle(\omega_{\xi,\eta} \otimes \mathrm{id}_{C(\mathbb{G})})U^x|\xi, \eta \in \mathcal{H}_x\rangle$. Then we have $\Delta : \mathcal{O}(\mathbb{G})_x \to \mathcal{O}(\mathbb{G})_x \odot \mathcal{O}(\mathbb{G})_x$ and $\mathcal{O}(\mathbb{G})_x^* = \mathcal{O}(\mathbb{G})_{\overline{x}}$. Moreover the matrixcoefficients u^x_{ij} form a linear basis of $\mathcal{O}(\mathbb{G})$.

Definition 2.2.13. *Let $\mathbb{G}$ be a compact quantum group. The reduced C^*-algebra $C_r(\mathbb{G})$ is defined as the norm closure of $\mathcal{O}(\mathbb{G})$ in the GNS-representation with respect to the Haar state h of $\mathbb{G}$.The universal C^*-algebra $C_u(\mathbb{G})$ is defined as the C^*-envelope of $\mathcal{O}(\mathbb{G})$.*

Note that if $\mathbb{G}$ is the dual of a discrete (classical) group Γ, we have $C_r(\mathbb{G}) = C_r^*(\Gamma)$ and $C_u(\mathbb{G}) = C_u^*(\Gamma)$.

Remark 2.2.14. *For a given compact quantum group $\mathbb{G}$, we have surjective morphisms between the the different completions of $\mathcal{O}(\mathbb{G})$: $C_u(\mathbb{G}) \to C(\mathbb{G}) \to C_r(\mathbb{G})$. We will think of all these algebras as describing the same quantum group. Therefore, we will identify all quantum groups having the same underlying Hopf *-algebra.*

Definition 2.2.15 ([79, 95]). *Let $\mathbb{G} = (C_u(\mathbb{G}), \Delta_{\mathbb{G}})$ and $\mathbb{H} = (C_u(\mathbb{H}), \Delta_{\mathbb{H}})$ be compact quantum groups equipped with their universal C^*-norms. Suppose moreover that there exists a surjective map $\theta : C_u(\mathbb{G}) \to C_u(\mathbb{H})$ satisfying $\Delta_{\mathbb{H}} \circ \theta = (\theta \otimes \theta)\Delta_{\mathbb{G}}$. Then we call $\mathbb{H}$ a quantum subgroup of $\mathbb{G}$. Equivalently, $\mathbb{G}$ is called a quantum supergroup of $\mathbb{H}$.*

2.3 Discrete quantum groups and duals of compact quantum groups

For compact groups, the Pontryagin dual of an abelian compact group is an abelian discrete group. This well known result can be lifted to the compact quantum group world. In this section, we recall the notions of discrete quantum groups and the dual of a compact quantum group. First we will fix some notations, agreeing with [93]. For an element i in an index set I let M_{n_i} be the set of n_i by n_i matrices. We say A is the direct sum over I of the algebras M_{n_i} if A consists of elements of the form $(x_i)_i$ where $x_i \in M_{n_i}$ for all $i \in I$ and where only finitely many x_i are not equal to zero. We will write $A = \oplus_{i\in I} M_{n_i}$ and see it as an algebra.
Moreover, we say B is the direct product of the algebras M_{n_i} if B consists of elements of the form $(x_i)_i$ where $x_i \in M_{n_i}$ for all $i \in I$. We will write $B = \prod_{i\in I} M_{n_i}$ and see it as the mutilplier algebra of $A = \oplus_{i\in I} M_{n_i}$. With these notations, one can state the following definitions. We begin with extending the notion of coproduct to algebras which might not have a unit.

Definition 2.3.1 ([92]). *Let A be an algebra with a non degenerate product and with or without unit. A coproduct or comultiplication is a homomorphism $\Delta : A \to \mathcal{M}(A \odot A)$ such that*

- $\Delta(a)(1 \otimes b)$ *and* $(a \otimes 1)\Delta(b)$ *are elements of* $A \odot A$ *for all* $a, b \in A$ *and*
- $(a \otimes 1 \otimes 1)(\Delta \otimes \mathrm{id})(\Delta(b)(1 \otimes c)) = (\mathrm{id} \otimes \Delta)((a \otimes 1)\Delta(b))(1 \otimes 1 \otimes c)$ *for* $a, b, c \in A$.

If A is a *-algebra, Δ is called a coproduct if it is also a *-morphism.

Definition 2.3.2 ([92]). *Let A be an algebra with a non degenerate product and with or without unit and Δ a comultiplication for A. Then A is called a multiplier Hopf algebra if the maps $T_1, T_2 : A \odot A \to A \odot A$, defined by*

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b) \qquad \text{and} \qquad T_2(a \otimes b) = (a \otimes 1)\Delta(b)$$

are bijective. If A is a *-algebra which satisfies the above conditions, it is called a multiplier Hopf *-algebra.

Definition 2.3.3 ([93]). *Let (A, Δ) be a pair where A is a direct sum of full matrix algebras and Δ is a comultiplication on A. If A has the structure of a multiplier Hopf *-algebra, A is called a discrete quantum group.*

Easy examples of discrete quantum groups are the algebras $C(\Gamma)$ where Γ is a discrete group with the coproduct coming from the product in Γ.

Definition 2.3.4 ([70]). *Let $\mathbb{G}$ be a compact quantum group. Denote*

$$c_0(\hat{\mathbb{G}}) = \oplus_{x \in \mathrm{Irred}(\mathbb{G})} B(\mathcal{H}_x),$$

$$\ell^\infty(\hat{\mathbb{G}}) = \prod_{x \in \mathrm{Irred}(\mathbb{G})} B(\mathcal{H}_x)$$

and

$$\mathbb{V} = \oplus_{x \in \mathrm{Irred}(\mathbb{G})} U^x \in \prod_{x \in \mathrm{Irred}(\mathbb{G})} B(\mathcal{H}_x) \otimes C(\mathbb{G}).$$

Now define

$$\hat{\Delta} : \ell^\infty(\hat{\mathbb{G}}) \to \ell^\infty(\hat{\mathbb{G}}) \overline{\otimes} \ell^\infty(\hat{\mathbb{G}}) : (\hat{\Delta} \otimes \mathrm{id})(\mathbb{V}) = \mathbb{V}_{13}\mathbb{V}_{23}.$$

Then $\hat{\Delta}_{|_{c_0(\hat{\mathbb{G}})}} : c_0(\hat{\mathbb{G}}) \to \ell^\infty(\hat{\mathbb{G}}) \overline{\otimes} \ell^\infty(\hat{\mathbb{G}})$ is a coproduct for $c_0(\hat{\mathbb{G}})$ and $\hat{\mathbb{G}} = (c_0(\hat{\mathbb{G}}), \hat{\Delta})$ is called the dual quantum group of $\mathbb{G}$. It has the structure of a discrete quantum group.

If needed, one can take the (unique) C^*-completion of $c_0(\hat{\mathbb{G}})$ to have a C^*-algebra $C(\hat{\mathbb{G}})$. As $C(\hat{\mathbb{G}}) \subset \ell^\infty(\hat{\mathbb{G}})$, one can restrict $\hat{\Delta}$ to $C(\hat{\mathbb{G}})$.

This equivalence generalizes the Pontryagin duality between compact abelian groups and discrete abelian groups.

2.4 Actions of compact quantum groups and the spectral subalgebra

Classically, groups appear very often in the context of group actions. In this section, we recapitulate the concept of (ergodic) actions of compact quantum groups and relate the actions to the unitary representations of the quantum group. For ergodic actions, one can prove the existence of a unique invariant state as well (as in the classical case).

Definition 2.4.1 ([79]). *Let B be a unital C^*-algebra and $\mathbb{G} = (C(\mathbb{G}), \Delta)$ a compact quantum group. A right action of $\mathbb{G}$ on B is a unital $*$-homomorphism $\beta : B \to B \otimes C(\mathbb{G})$ such that*

1. $(\beta \otimes \mathrm{id}_{C(\mathbb{G})})\beta = (\mathrm{id}_B \otimes \Delta)\beta$
2. $[\beta(B)(1 \otimes C(\mathbb{G}))] = B \otimes C(\mathbb{G})$.

Analogously, a left action is a unital *-morphism $\beta' : B \to C(\mathbb{G}) \otimes B$ satisfying the analogous conditions. We say that the action is ergodic if $B^\beta = \{b \in B | \beta(b) = b \otimes 1\} = \mathbb{C}1_B$.

One can choose to call the map in this definition 'a coaction' as it is a coaction of the C^*-algebra $C(\mathbb{G})$ on B. However, we choose to call it an action of the compact quantum group in order to be compatible with the classical case: if $C(\mathbb{G}) = C(G)$ and $B = C(X)$ with G a classical compact group and X a compact space, it is an action of G on X.

Proposition 2.4.2 ([28]). *Let β be a right ergodic action of a compact quantum group $\mathbb{G}$ on a unital C^*-algebra B. Then B admits a unique invariant state ω.*

Proof. Define the map $E : B \to B : b \mapsto (\mathrm{id}_B \otimes h)\beta(b)$ where h is the Haar state. We claim that $E(B) \subset \mathbb{C}1_B$. Indeed, we have

$$\begin{aligned}
\beta(E(b)) &= \beta(\mathrm{id}_B \otimes h)\beta(b) \\
&= (\mathrm{id}_B \otimes \mathrm{id}_{C(\mathbb{G})} \otimes h)(\beta \otimes \mathrm{id}_{C(\mathbb{G})})\beta(b) \\
&= (\mathrm{id}_B \otimes \mathrm{id}_{C(\mathbb{G})} \otimes h)(\mathrm{id}_B \otimes \Delta)\beta(b) \\
&= (\mathrm{id}_B \otimes h)\beta(b) \otimes 1 \\
&= E(b) \otimes 1.
\end{aligned}$$

As β is ergodic, this implies that $E(b) \in \mathbb{C}1_B$. Therefore, we can define a state ω such that $\omega(b)1_B = E(b)$ for every $b \in B$. Moreover, ω is invariant under β:

$$\begin{aligned}
(E \otimes \mathrm{id}_{C(\mathbb{G})})\beta(b) &= (\mathrm{id}_B \otimes h \otimes \mathrm{id}_{C(\mathbb{G})})(\beta \otimes \mathrm{id}_{C(\mathbb{G})})\beta(b) \\
&= (\mathrm{id}_B \otimes h \otimes \mathrm{id}_{C(\mathbb{G})})(\mathrm{id}_B \otimes \Delta)\beta(b) \\
&= E(b) \otimes 1
\end{aligned}$$

and hence $(\omega \otimes \mathrm{id}_{C(\mathbb{G})})\beta(b) = \omega(b)1_{C(\mathbb{G})}$. Finally, suppose there is another invariant state ω', then we have

$$\omega(b) = \omega'(\omega(b)1_B) = (\omega' \otimes h)\beta(b) = h(\omega' \otimes \mathrm{id})\beta(b) = h(\omega'(b)1_{C(\mathbb{G})}) = \omega'(b).$$

This proves the statement. $\square$

Note that the most evident example of an action is a compact quantum group acting on itself by comultiplication. Moreover, in this case the action is ergodic and one can check that $\omega = h$.

Proposition 2.4.3. *Let β_1 resp. β_2, be a left resp. right ergodic action of a compact quantum group $\mathbb{G}$ on a C^*-algebra B such that β_1 and β_2 commute. Denote by ω the unique β_2-invariant state on B such that $\omega(b)1_B = (\mathrm{id} \otimes h)\beta_2(b)$. Then $\omega(b)1_B = (h \otimes \mathrm{id})\beta_1(b)$.*

Proof. Denote for $b \in B$ $E_1 : B \to B : b \to (h \otimes \mathrm{id})\beta_1(b)$, then we have

$$\begin{aligned}
\beta_1(E_1(b)) &= \beta_1(h \otimes \mathrm{id}_B)\beta_1(b) \\
&= (h \otimes \mathrm{id}_{C(\mathbb{G})} \otimes \mathrm{id}_B)(\mathrm{id}_{C(\mathbb{G})} \otimes \beta_1)\beta_1(b) \\
&= (h \otimes \mathrm{id}_{C(\mathbb{G})} \otimes \mathrm{id}_B)(\Delta \otimes \mathrm{id}_B)\beta_1(b) \\
&= 1_{C(\mathbb{G})} \otimes (h \otimes \mathrm{id}_B)\beta_1(b) \\
&= 1_{C(\mathbb{G})} \otimes E_1(b)
\end{aligned}$$

and as β_1 is ergodic, there exists a state ω_1 such that $E_1(b) = \omega_1(b)1_B$ for every $b \in B$. Analogously as in the proof of proposition 2.4.2, ω_1 is $\beta_1$invariant. Now, as β_1 and β_2 commute, we have

$$\begin{aligned}
(E_1 \otimes \mathrm{id}_{C(\mathbb{G})})\beta_2(b) &= (h \otimes \mathrm{id}_B \otimes \mathrm{id}_{C(\mathbb{G})})(\beta_1 \otimes \mathrm{id})\beta_2(b) \\
&= (h \otimes \mathrm{id}_B \otimes \mathrm{id}_{C(\mathbb{G})})(\mathrm{id}_{C(\mathbb{G})} \otimes \beta_2)\beta_1(b) \\
&= \beta_2(h \otimes \mathrm{id}_B)\beta_1(b) \\
&= \beta_2(\omega_1(b)1_B) \\
&= E_1(b) \otimes 1_{C(\mathbb{G})}
\end{aligned}$$

which implies that ω_1 is also β_2-invariant. As ω was the unique β_2-invariant state, we have $\omega_1 = \omega$. $\square$

In what follows, we will investigate the intimate link between an ergodic action of a compact quantum group on a unital C^*-algebra and the representations of the quantum group.

In [28] the decomposition of a right action into right irreducible representations is described. This is clear: as we have the invariant state ω, we can make the GNS representation space $\mathcal{H}_\omega$ of B and look at how the representation of $\mathbb{G}$ on $\mathcal{H}_\omega$ split up into irreducible representations. However, here we relate right representations to left actions $B \to C(\mathbb{G}) \otimes B$. In that sense, what follows is new work, but build on results of [28].

Definition 2.4.4. [1] *Let $\mathbb{G}$ be a compact quantum group and β a left ergodic action on a unital C^*-algebra B. Let $x \in \mathrm{Irred}(\mathbb{G})$. We define the spectral subspace corresponding to x to be the set*

$$K_x = \{\zeta \in \mathcal{H}_x \odot B | U^x_{12}\zeta_{13} = (\mathrm{id}_{\mathcal{H}_x} \otimes \beta)\zeta\}.$$

With the notations of chapter 1, one sees that $K_x = \mathcal{H}_x \underset{\mathcal{O}(\mathbb{G})}{\boxdot} B$. Via K_x one can define the following subspaces of B:

Definition 2.4.5. [2] *For $x \in \mathrm{Irred}(\mathbb{G})$, we define $\mathcal{B}_x$ to be*

$$\mathcal{B}_x = \langle (\xi^* \otimes 1_B)\zeta | \zeta \in K_x, \xi \in \mathcal{H}_x \rangle.$$

Then we have $\beta : \mathcal{B}_x \to \mathcal{O}(\mathbb{G})_x \odot \mathcal{B}_x$ and $\mathcal{B}_x^* = \mathcal{B}_{\overline{x}}$.

Moreover we define $\mathcal{B}$ to be the subspace of B generated by all the K_x:

$$\mathcal{B} = \langle (\xi^* \otimes 1_B)\zeta | x \in \mathrm{Irred}(\mathbb{G}), \zeta \in K_x, \xi \in \mathcal{H}_x \rangle.$$

Now, we prove that K_x is finite dimensional and we will put a Hilbert space structure on it.

Proposition 2.4.6.

- *K_x is finite dimensional for every $x \in \mathrm{Irred}(\mathbb{G})$.*
- *For $\zeta_1, \zeta_2 \in K_x$ and denoting $\zeta^* = \sum_i \xi_i^* \otimes b_i^* \in \overline{\mathcal{H}_x} \otimes B$ if $\zeta = \sum_i \xi_i \otimes b_i$ (with $(\xi_i)_i$ an orthonormal basis of $\mathcal{H}_x$), we have $\zeta_1^*\zeta_2 \in \mathbb{C}1_B$.*
- *K_x is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ such that*

$$\langle \zeta_1, \zeta_2 \rangle 1 = \zeta_1^* \zeta_2.$$

Proof.

- One can easily see that $K_x \subset \mathcal{H}_x \odot \mathcal{B}_x$ and it follows from [28], that $\mathcal{B}_x$ is finite dimensional. As also $\mathcal{H}_x$ is finite dimensional, the statement is proven.
- We have

$$\beta(\zeta_1^*\zeta_2) = (\zeta_1)_{13}^*(U^x)_{12}^* U^x_{12}(\zeta_2)_{13} = (\zeta_1)_{13}^*(\zeta_2)_{13} = 1_B \otimes \zeta_1^*\zeta_2$$

implying that $\zeta_1^*\zeta_2 \in \mathbb{C}1_B$ as β is ergodic.

[1] One can prove easily that the notion of spectral subspaces found in literature (e.g. [28]) is in bijective correspondence with our notion. Indeed, one finds that for a class y of irreducible unitary left representations $K'_y = \{Y \in B \odot \overline{H_y} | (\beta_1 \otimes \mathrm{id})Y = Y_{23}U^y_{13} \in C(\mathbb{G}) \otimes B\}$ is called the spectral subspace in literature. One can prove that $K_{\overline{x}} \cong K'_{x'}$ for $x \in \mathrm{Irred}(\mathbb{G})$ where $U^{x'} = \sigma U^x$.

[2] One can prove that for $\mathcal{B}'_y = \langle Y(1_B \otimes \xi) | Y \in K'_y, \xi \in \mathcal{H}_y \rangle$, found in literature [28], one has $\mathcal{B}_{\overline{x}} \cong \mathcal{B}'_{x'}$, where again, $U^{x'} = \sigma U_x$.

- Linearity in the second and antilinearity in the first argument can be seen directly. Moreover $\overline{\langle\zeta_1,\zeta_2\rangle} = \langle\zeta_2,\zeta_1\rangle$ as $(\zeta_1^*\zeta_2)^* = \zeta_2^*\zeta_1$ and if $\zeta = \sum \xi_i \otimes b_i$ (with $(\xi_i)_i$ an orthonormal basis of $\mathcal{H}_x$) then $\langle\zeta,\zeta\rangle = \sum b_i^* b_i \geq 0$. Moreover, if $\langle\zeta,\zeta\rangle = 0$, then $b_i^* b_i = 0$ and hence $b_i = 0$ for all i and therefore $\zeta = 0$. By construction it is complete as metric space.

□

With $\mathcal{B}$ defined as in definition 2.4.5, one can state the following important proposition.

Proposition 2.4.7 ([28]). [3] *Let B be a unital C^*-algebra and $\beta : B \to B \otimes C(\mathbb{G})$ an ergodic action of $\mathbb{G}$ on B. Then $\mathcal{B}$ is a dense unital *-subalgebra of B which we will call the spectral subalgebra of B with respect to β. Moreover $\beta_{|\mathcal{B}}$ is an algebraic coaction of the Hopf *-algebra $(\mathcal{O}(\mathbb{G}),\Delta)$ on $\mathcal{B}$.*

This proposition will be one of the tools to link (analytical) properties of an action $\beta : B \to B \otimes C(\mathbb{G})$ to (algebraic) properties of the coaction $\beta' : \mathcal{B} \to \mathcal{B} \odot \mathcal{O}(\mathbb{G})$.

Remark 2.4.8. *An action $\beta : B \to B \otimes C(\mathbb{G})$ of $\mathbb{G}$ on B is called universal if B is the universal C^*-algebra of $\mathcal{B}$. It is called reduced if the map $(\mathrm{id} \otimes h)\beta : B \to B$ onto the fixed point algebra B^β is faithful.*
Note that in remark 2.2.14, we saw that a compact quantum group can be described using different C^*-versions with the same underlying (dense) Hopf *-subalgebra.Similarly here: we have surjective morphisms: $B_u \to B \to B_r$ for an action $\beta : B \to B \otimes C(\mathbb{G})$ of $\mathbb{G}$ on B. So again, we will identify two actions if the underlying Hopf *-algebras are the same.

2.5 Actions of full quantum multiplicity

In section 1.2, we recalled that ergodic (algebraic) coactions can have the extra property to have a bijective Galois map. In this section, we describe the analytical counterpart of this property, which will be called action of full quantum multiplicity. This notion was defined in [27]. Furthermore it makes a very important link with monoidal equivalences (which we will introduce in the next section). As in section 2.4, we make left-right adaptations of the work of [27] as we continue to work with the spaces K_x (definition 2.4.4) which were defined differently as in [27].

[3] With the remarks in the previous footnotes, it is easy to see that the notion of spectral subalgebra here coincides with the notion of spectral subalgebra in literature.

First, define, for every $x \in \mathrm{Irred}(\mathbb{G})$, $X^x \in \mathcal{B}(K_x, \mathcal{H}_x) \otimes B$ by

$$(X^x)(\zeta \otimes 1) = \zeta \tag{2.5.1}$$

for $\zeta \in K_x$. Note that we will write $\overline{\zeta}$ as an element in $\overline{K}_x$ and ζ^* as element of $\overline{\mathcal{H}}_x \otimes B$. With this convention, we also have $(\overline{\zeta} \otimes 1)(X^x)^* = \zeta^*$.

Proposition 2.5.1. *Let $x \in \mathrm{Irred}(G)$ and X^x as defined above. Then $(X^x)^* X^x = 1$ and hence $X^x (X^x)^*$ is a projection.*

Proof. Let's define some notation first. Let $(\xi_i^x)_i, (\zeta_j^x)_j$ to be orthonormal basises of $\mathcal{H}_x, K_x$ respectively. As K_x is a sub vector space of $\mathcal{H}_x \otimes B$, we can find elements b_{ij}^x such that $\zeta_j^x = \sum_i \xi_i^x \otimes b_{ij}^x$. (To avoid too much notation, we will write ζ_j, ξ_i, b_{ij} for $\zeta_j^x, \xi_i^x, b_{ij}^x$, when the irreducible representation x is clear.) Writing $X^x = \sum_{s,t} \xi_s \overline{\zeta}_t \otimes b'_{st}$, we get

$$(X^x)(\zeta_j \otimes 1) = \sum_s \xi_s \otimes b'_{sj} \tag{2.5.2}$$

and as $(X^x)(\zeta_j \otimes 1) = \zeta_j$, we can conclude that $b'_{ij} = b_{ij}$.

Moreover, writing $U^x(\xi_t \otimes \mathrm{id}) = \sum_s \xi_s \otimes u_{st}$, we have, for all j, $(\mathrm{id}_{\mathcal{H}_x} \otimes \beta)(\zeta_j) = \sum_i \xi_i \otimes \beta(b_{ij})$ and

$$U_{12}^x(\zeta_j)_{13} = U_{12}^x(\sum_k \xi_k \otimes 1 \otimes b_{kj}) = \sum_{i,k} \xi_i \otimes u_{ik} \otimes b_{kj} \tag{2.5.3}$$

and we can conclude that

$$\beta(b_{ij}) = \sum_k u_{ik} \otimes b_{kj} \tag{2.5.4}$$

for every i, j. Furthermore, as $(\zeta_j)_j$ is an orthonormal basis of K_x, we have

$$\delta_{i,j} = \langle \zeta_i, \zeta_j \rangle = \langle \sum_k \xi_k \otimes b_{ki}, \sum_l \xi_l \otimes b_{lj} \rangle = \sum_k b_{ki}^* b_{kj}. \tag{2.5.5}$$

Therefore we have:

$$\begin{aligned}
(X^x)^* X^x &= \sum_{s,s',t,t'} \zeta_t \xi_s^* (\xi_{s'}) \overline{\zeta}_{t'} \otimes b_{st}^* b_{s't'} \\
&= \sum_{s,t',t} \zeta_t \overline{\zeta}_{t'} \otimes b_{st}^* b_{st'}
\end{aligned}$$

$$\begin{aligned} &= \sum_{t',t} \zeta_t \overline{\zeta}_{t'} \otimes \delta_{t,t'} \\ &= \sum_{t} \zeta_t \overline{\zeta}_{t} \\ &= 1 \end{aligned}$$

The fact that $X^x(X^x)^*$ is a projection, follows then immediately. □

In definition 2.2.10 we introduced the notion of the quantum dimension of an irreducible representation as a new way of measuring a dimension. Here we will introduce the notion of quantum multiplicity as a new notion of multiplicity of a representation in an action.

So, let $\beta : B \to B \otimes C(\mathbb{G})$ be an ergodic action and x an element in $\mathrm{Irred}(\mathbb{G})$. Taking $t_x \in \mathrm{Mor}(x \otimes \overline{x}, \varepsilon)$ such that $t_x^* t_x = \dim_q(x)$, we can define:

$$R_x : K_x \to K_{\overline{x}} : \zeta \mapsto \zeta_{13}^*(t_x \otimes \mathrm{id}_B).$$

Indeed, as $U_{13}^x U_{23}^{\overline{x}}(t_x \otimes 1_{C(\mathbb{G})}) = t_x \otimes 1_{C(\mathbb{G})}$, we have

$$\begin{aligned} (\mathrm{id}_{\mathcal{H}_{\overline{x}}} \otimes \beta)(\zeta_{13}^*(t_x \otimes \mathrm{id}_B)) &= (\mathrm{id}_{\mathcal{H}_{\overline{x}}} \otimes \mathrm{id}_{\mathcal{H}_x} \otimes \beta)(\zeta_{13}^*)(t_x \otimes \mathrm{id}_{C(\mathbb{G})} \otimes \mathrm{id}_B) \\ &= \zeta_{14}^*(U^x)_{13}^*(t_x \otimes \mathrm{id}_{C(\mathbb{G})} \otimes \mathrm{id}_B) \\ &= \zeta_{14}^* U_{23}^{\overline{x}}(t_x \otimes \mathrm{id}_{C(\mathbb{G})} \otimes \mathrm{id}_B) \\ &= U_{23}^{\overline{x}} \zeta_{14}^*(t_x \otimes \mathrm{id}_{C(\mathbb{G})} \otimes \mathrm{id}_B) \\ &= U_{12}^{\overline{x}} \Big(\zeta_{13}^*(t_x \otimes \mathrm{id}_B)\Big)_{13} \end{aligned}$$

and hence R_x is well defined.

Since t_x is normalized up to a phase of modulus one, $L_x = R_x^* R_x$ is a well defined positive bounded operator on K_x.

Definition 2.5.2. *We define* $\mathrm{mult}_q(x) = \sqrt{\mathrm{Tr}(L_x)\,\mathrm{Tr}(L_{\overline{x}})}$ *and call it the quantum multiplicity of* x *in* β*.*

Note that, by definition, $\mathrm{mult}_q(x) = \mathrm{mult}_q(\overline{x})$. Also, it is easy to check that $R_{\overline{x}} = R_x^{-1}$ and hence $L_{\overline{x}} = (R_x R_x^*)^{-1}$ and $\mathrm{Tr}(L_{\overline{x}}) = \mathrm{Tr}(L_x^{-1})$. So $\mathrm{mult}_q(x) = \sqrt{\mathrm{Tr}(L_x)\,\mathrm{Tr}(L_x^{-1})}$.

Proposition 2.5.3 ([27,28,67]). *Let $\beta : B \to B \otimes C(\mathbb{G})$ be an ergodic action of a compact quantum group $\mathbb{G}$ on a unital C^*-algebra B. Then*

$$\dim_q(x) \geq \mathrm{mult}_q(x) \geq \mathrm{mult}(x)$$

for all $x \in \mathrm{Irred}(\mathbb{G})$. If $\dim_q(x) = \mathrm{mult}_q(x)$ holds for every $x \in \mathrm{Irred}(\mathbb{G})$, we call β of full quantum multiplicity. Moreover, β is of full quantum multiplicity if and only if X^x as defined above, is unitary.

Proof. We slightly adapt the proof of theorem 2.9 in [27] as there, one works with a right action of $\mathbb{G}$ on B and hence a slightly different definition of K_x. Note first that

$$\mathrm{mult}_q(x) = \sqrt{\mathrm{Tr}(L_x)\,\mathrm{Tr}(L_x^{-1})} \geq \dim(K_x) = \mathrm{mult}(x)$$

as the trace of $L_x \otimes L_x^{-1}$ is bigger than $\dim(K_x)^2$ for all $x \in \mathrm{Irred}(\mathbb{G})$. Now note that, by definition:

$$\begin{aligned}
\langle \zeta_j, L_x \zeta_j \rangle &= \langle R_x \zeta_j, R_x \zeta_j \rangle \\
&= \langle (\zeta_j^*)_{13}(t_x \otimes \mathrm{id}_B), (\zeta_j^*)_{13}(t_x \otimes \mathrm{id}_B) \\
&= (t_x \otimes \mathrm{id}_B)^* (\zeta_j \zeta_j^*)_{13}(t_x \otimes \mathrm{id}_B).
\end{aligned}$$

Now one can define, analogous to the maps j_x in (2.2.1), the following anti linear maps:

$$s_x : \mathcal{H}_{\overline{x}} \to \mathcal{H}_x : \eta \mapsto (\mathrm{id}_{\mathcal{H}_x} \otimes \eta^*) t_x.$$

One can easily check that then $t_x^*(a \otimes 1)t_x = \mathrm{Tr}(S_x a)$ for $a \in B(\mathcal{H}_x)$ where $S_x = s_x s_x^* : \mathcal{H}_x \to \mathcal{H}_x$. Moreover, $\mathrm{Tr}(S_x) = \mathrm{Tr}(Q_x)$. Hence we have

$$\begin{aligned}
\mathrm{Tr}(L_x)1 &= \sum_{j=1}^{\dim K_x} (\mathrm{Tr}(S_x \cdot) \otimes \mathrm{id})(\zeta_j \zeta_j^*) \\
&= \sum_{j=1}^{\dim K_x} (\mathrm{Tr}(S_x \cdot) \otimes \mathrm{id})\Big(X^x (\zeta_j \overline{\zeta}_j \otimes \mathrm{id})(X^x)^* \Big) \\
&= (\mathrm{Tr}(S_x \cdot) \otimes \mathrm{id})(X^x (X^x)^*).
\end{aligned}$$

As $X^x(X^x)^*$ is a projection, $(\mathrm{Tr}(S_x \cdot) \otimes \mathrm{id})(X^x(X^x)^*) \leq \mathrm{Tr}(S_x)1 = \mathrm{Tr}(Q_x)1$ and we can conclude that $\mathrm{mult}_q(x) \leq \dim_q(x)$. Moreover, β is of full quantum multiplicity if and only if

$$(\mathrm{Tr}(S_x \cdot) \otimes \mathrm{id})(X^x(X^x)^*) = \mathrm{Tr}(L_x)1 = \mathrm{Tr}(Q_x)1 = \mathrm{Tr}(S_x)1$$

which holds if and only if $X^x(X^x)^* = 1$ and hence if and only if X^x is unitary. $\square$

This proposition is the last puzzle piece to prove the intimate link between actions of full quantum multiplicity and Galois objects. The two concepts are stated in a different context but turn out to be equivalent. This result was known but never explicitly proven. This is the goal of the following theorem:

Theorem 2.5.4. *Let $\beta : B \to C(\mathbb{G}) \otimes B$ be an ergodic action of a compact quantum group $\mathbb{G}$ on a unital C^*-algebra B. Denote by $\beta' : \mathcal{B} \to \mathcal{O}(\mathbb{G}) \odot \mathcal{B}$ the associated coaction of $\mathcal{O}(\mathbb{G})$ on $\mathcal{B}$. Let X^x as defined in the beginning of this section and $T_{\beta'}$ as in the first chapter. Then the following are equivalent:*

1. *β is of full quantum multiplicity*
2. *X^x is unitary for all irreducible representations x.*
3. *$T_{\beta'}$ is a Galois map and hence $\mathcal{B}$ is a left $\mathcal{O}(\mathbb{G})$-Galois object.*

Proof. **1 ⇔ 2** Is proven in proposition 2.5.3.

2 ⇔ 3 Suppose first that X^x is unitary for every irreducible representation x. Writing $X^x = \sum_{s,t} \xi_s \overline{\zeta}_t \otimes b_{st}$ as in the proof of proposition 2.5.1, unitarity of X^x implies that $X^x(X^x)^* = 1$ and hence

$$1 = X^x(X^x)^* = \sum_{s,s',t,t'} \xi_s \overline{\zeta}_t(\zeta_{t'})\xi_{s'}^* \otimes b_{st} b_{s't'}^* = \sum_{s,s',t} \xi_s \xi_{s'}^* \otimes b_{st} b_{s't}^*$$

which implies $\sum_t b_{st} b_{s't}^* = \delta_{s,s'}$. Now define the following map:

$$\gamma : \mathcal{O}(\mathbb{G}) \to \mathcal{B} \odot \mathcal{B} : u_{ik}^x \mapsto \sum_j b_{ij}^x \otimes (b_{kj}^x)^*,$$

then, we have $T_\beta(\gamma(a)) = a \otimes 1$. Indeed, we have:

$$T_\beta(\gamma(u_{ik}^x)) = T_\beta\Big(\sum_j b_{ij}^x \otimes (b_{kj}^x)^*\Big) = \sum_{j,l} u_{il}^x \otimes b_{lj}^x (b_{kj}^x)^* = u_{ik}^x \otimes 1$$

where we used $\beta(b_{ik}^x) = \sum_j u_{ij}^x \otimes b_{jk}^x$ which we found in equation (2.5.4). Extending this, we find a map

$$P_\beta : \mathcal{O}(\mathbb{G}) \odot \mathcal{B} \to \mathcal{B} \odot \mathcal{B} : a \otimes b \to \gamma(a)(1 \otimes b)$$

which is an inverse for T_β. Indeed we have:

$$T_\beta(P_\beta(u_{ij}^x \otimes b_{st}^y)) = T_\beta(\sum_k b_{ik}^x \otimes (b_{jk}^x)^* b_{st}^y) = \sum_{k,l} u_{il}^x \otimes b_{lk}^x (b_{jk}^x)^* b_{st}^y$$

$$= u^x_{ij} \otimes b^y_{st} \tag{2.5.6}$$

and also

$$P_\beta(T_\beta(b^x_{ij} \otimes b^y_{st})) = P_\beta(\sum_k u^x_{ik} \otimes b^x_{kj} b^y_{st}) = \sum_{k,l} b^x_{il} \otimes (b^x_{kl})^* b^x_{kj} b^y_{st}$$

$$= b^x_{ij} \otimes b^y_{st} \tag{2.5.7}$$

where we used $\sum_k (b^x_{kl})^* b^x_{kj} = \delta_{j,l}$ obtained in the proof of proposition 2.5.1. Hence we proved that T_β is a bijection and hence a Galois map. Conversely, suppose first that T_β is bijective. Using the maps γ and P_β as before, we see that the equation 2.5.7 does not use the requirement that X^x is unitary. Hence it also holds without assuming X^x is unitary. This implies however that P_β is indeed the inverse of T_β. Using that $\varepsilon = m \circ \gamma$, we have

$$\delta_{ij} = \varepsilon(u^x_{ij}) = m \circ \gamma(u^x_{ij}) = m(\sum_k b^x_{ik} \otimes (b^x_{jk})^*) = \sum_k b^x_{ik}(b^x_{jk})^*$$

for every $x \in \mathrm{Irred}(\mathbb{G})$ which implies that X^x is unitary for every $x \in \mathrm{Irred}(\mathbb{G})$.

$\square$

2.6 Monoidal equivalences between compact quantum groups

In the previous section we explored the meaning of actions of full quantum multiplicity and proved that this notion is equivalent to the coactions on Galois objects. In chapter 1 we proved that a Galois object induces an equivalence of the categories of comodules. In this analytical approach, we have this equivalence relation as well.

Definition 2.6.1 ([27]). *Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group. A unitary fiber functor is a collection of maps ψ such that*

- *for every $x \in \mathrm{Irred}(\mathbb{G})$, there is a finite dimensional Hilbert space $\mathcal{H}_{\psi(x)}$,*
- *there are linear maps*

$$\psi : \mathrm{Mor}(x_1 \otimes \ldots \otimes x_k, y_1 \otimes \ldots \otimes y_s)$$
$$\to B(\mathcal{H}_{\psi(y_1)} \otimes \ldots \otimes \mathcal{H}_{\psi(y_s)}, \mathcal{H}_{\psi(x_1)} \otimes \ldots \otimes \mathcal{H}_{\psi(x_k)}) \tag{2.6.1}$$

satisfying

$$\psi(1) = 1, \qquad \psi(S \otimes T) = \psi(S) \otimes \psi(T),$$
$$\psi(S^*) = \psi(S)^*, \qquad \psi(ST) = \psi(S)\psi(T) \tag{2.6.2}$$

whenever the formulas make sense.

Remark 2.6.2 ([27]). *To define a unitary fiber functor it suffices to attach to every $x \in \mathrm{Irred}(\mathbb{G})$ a finite dimensional Hilbert space $\mathcal{H}_{\psi(x)}(\mathcal{H}_\varepsilon = \mathbb{C})$ and to define the linear maps*

$$\psi : \mathrm{Mor}(x_1 \otimes \ldots \otimes x_k, y) \to B(\mathcal{H}_{\psi(y)}, \mathcal{H}_{\psi(x_1)} \otimes \ldots \otimes \mathcal{H}_{\psi(x_k)})$$

for $k = 1, 2, 3$ satisfying

$$\begin{aligned} \psi(1) &= 1 \\ \psi(S)^*\psi(T) &= \psi(S^*T) \quad \textit{if} \quad S \in \mathrm{Mor}(x \otimes y, a), T \in \mathrm{Mor}(x \otimes y, b) \\ (\psi(S) \otimes \mathrm{id})\psi(T) &= \psi((S \otimes \mathrm{id})T) \quad \textit{if} \quad S \in \mathrm{Mor}(x \otimes y, a), T \in \mathrm{Mor}(a \otimes z, b) \\ (\mathrm{id} \otimes \psi(S))\psi(T) &= \psi((\mathrm{id} \otimes S)T) \quad \textit{if} \quad S \in \mathrm{Mor}(y \otimes z, a), T \in \mathrm{Mor}(x \otimes a, b) \end{aligned} \tag{2.6.3}$$

together with a non-degenerateness condition

$$[\psi(S)\xi | x \in \mathrm{Irred}(\mathbb{G}), S \in \mathrm{Mor}(y \otimes z, x), \xi \in \mathcal{H}_{\psi(x)}] = \mathcal{H}_{\psi(y)} \otimes \mathcal{H}_{\psi(z)}$$

for $y, z \in \mathrm{Irred}(\mathbb{G})$.

Definition 2.6.3 ([27]). *Let $\mathbb{G}_1 = (C(\mathbb{G}_1), \Delta_1)$ and $\mathbb{G}_2 = (C(\mathbb{G}_2), \Delta_2)$ be two compact quantum groups. $\mathbb{G}_1$ and $\mathbb{G}_2$ are called monoidally equivalent if there exists a bijection $\varphi : \mathrm{Irred}(\mathbb{G}_1) \to \mathrm{Irred}(\mathbb{G}_2)$ which satisfies $\varphi(\varepsilon_{\mathbb{G}_1}) = \varepsilon_{\mathbb{G}_2}$ together with linear isomorphisms:*

$$\begin{aligned} \varphi : \mathrm{Mor}&\left(x_1 \otimes \ldots \otimes x_r, y_1 \otimes \ldots \otimes y_k\right) \\ &\to \mathrm{Mor}\left(\varphi(x_1) \otimes \ldots \otimes \varphi(x_r), \varphi(y_1) \otimes \ldots \otimes \varphi(y_k)\right) \end{aligned}$$

which satisfy the equations (2.6.2) *of definition 2.6.1. The collection of maps is called a monoidal equivalence.*

Note that this is indeed the usual definition of equivalence between strict monoidal categories, but adapted to the concrete case of the category of representations of a compact quantum group. Moreover, a monoidal equivalence φ implies that $\dim_q(\varphi(x)) = \dim_q(x)$ for all $x \in \mathrm{Irred}(\mathbb{G})$. Indeed, applying (2.6.2) to $t_x \in \mathrm{Mor}(x \otimes \overline{x}, \varepsilon)$, we have

$$\dim_q(\varphi(x))1 = \varphi(t_x)^*\varphi(t_x) = \varphi(t_x^* t_x) = \varphi(1)\dim_q(x) = \dim_q(x)1.$$

In fact, the notions of unitary fiber functor and monoidal equivalence are equivalent, which is stated in the following proposition, taken from Proposition 3.12 in [27]. In some sense this is an analogue of theorem 1.2.8: information about one compact quantum group implies the existence of another CQG which is equivalent to the first.

Proposition 2.6.4. *Let $\mathbb{G}_1$ be a compact quantum group and ψ a unitary fiber functor on it. Then there exist a unique universal compact quantum group $\mathbb{G}_2$ with underlying Hopf algebra $(\mathcal{O}(\mathbb{G}_2), \Delta_2)$ and unitary representations $U^{\psi(x)} \in B(\mathcal{H}_{\psi(x)}) \otimes C(\mathbb{G}_2)$, $x \in \mathrm{Irred}(\mathbb{G}_1)$ such that*

1. *$U_{13}^{\psi(y)} U_{23}^{\psi(z)}(\psi(S) \otimes 1) = (\psi(S) \otimes 1)U^{\psi(x)}$ for all $S \in \mathrm{Mor}(y \otimes z, x)$,*
2. *the matrix coefficients of the $U^{\psi(x)}$, $x \in \mathrm{Irred}(\mathbb{G}_1)$ form a linear basis of $\mathcal{O}(\mathbb{G}_2)$.*

Moreover, the set $\{U^{\psi(x)} | x \in \mathbb{G}_1\}$ forms a complete set of irreducible representations of $\mathbb{G}_2$ and the unitary fiber functor ψ on $\mathbb{G}_1$ induces a monoidal equivalence $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$.

The following theorems of Bichon et al. will be crucial for our deformation procedure. They explain what extra structure a monoidal equivalence induces.

The first theorem follows from Theorem 3.9 and Proposition 3.13 of [27].

Theorem 2.6.5 ([27]). [4] *Let $\mathbb{G}_1$ be a compact quantum group and let ψ be a unitary fiber functor on $\mathbb{G}_1$. Denote with $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$ the monoidal equivalence induced by ψ (see previous proposition).*

[4] In the original statement of [27] the coaction β_1 is a right coaction of $C(\mathbb{G}_1)$, but for what follows, we want a left coaction of $C(\mathbb{G}_1)$ and a right coaction of $C(\mathbb{G}_2)$. Applying Bichon's theorem on the inverse monoidal equivalence $\varphi' : \mathbb{G}_2 \to \mathbb{G}_1$, one gets the theorem stated here. Note that, when doing that, we should write $X^{\varphi(x)}$, $x \in \mathrm{Irred}(\mathbb{G}_1)$ but for notational convenience, we write $X^x, x \in \mathrm{Irred}(\mathbb{G}_1)$.

1. *There exists a unique unital *-algebra $\mathcal{B}$ equipped with a unique faithful state ω and unitary elements $X^x \in B(\mathcal{H}_{\varphi(x)}, \mathcal{H}_x) \odot \mathcal{B}$ for all $x \in \mathrm{Irred}(\mathbb{G}_1)$ satisfying*
   (a) $X^y_{13} X^z_{23}(\varphi(S) \otimes 1) = (S \otimes 1)X^x$ *for all* $S \in \mathrm{Mor}(y \otimes z, x)$,
   (b) *the matrix coefficients of the* X^x *form a linear basis of* $\mathcal{B}$,
   (c) $(\mathrm{id} \otimes \omega)(X^x) = 0$ *if* $x \neq \varepsilon$.
2. *There exist unique commuting coactions $\beta_1 : \mathcal{B} \to \mathcal{O}(\mathbb{G}_1) \odot \mathcal{B}$ and $\beta_2 : \mathcal{B} \to \mathcal{B} \odot \mathcal{O}(\mathbb{G}_2)$ satisfying*
$$(\mathrm{id} \otimes \beta_1)(X^x) = U^x_{12} X^x_{13} \qquad \textit{and} \qquad (\mathrm{id} \otimes \beta_2)(X^x) = X^x_{12} U^{\varphi(x)}_{13}$$
for all $x \in \mathrm{Irred}(\mathbb{G}_1)$. Moreover, $\omega(b)1_B = (h \otimes \mathrm{id}_B)\beta_1(b)$.
3. *The state ω is invariant under β_1 and β_2. Denoting by B_r the C^*-algebra generated by $\mathcal{B}$ in the GNS-representation associated with ω and denoting by B_u the universal enveloping C^*-algebra of $\mathcal{B}$, the Hopf algebraic coactions β_1 and β_2 admit unique extensions to actions of the compact quantum groups on B_r, resp. B_u. These actions are reduced, resp. universal and they are ergodic and of full quantum multiplicity.*
4. *Every reduced, resp. universal, ergodic action of full quantum multiplicity, arises in this way from a monoidal equivalence.*

Definition 2.6.6. *In what follows, we will call $\mathcal{B}$ the ($\mathbb{G}_1$-$\mathbb{G}_2$)-bi-Galois object associated with φ.*

Using the parts three and four of theorem 2.6.5, we can 'upgrade' theorem 2.5.4. This gives us a clear description of seemingly different but equivalent concepts:

Theorem 2.6.7. *Let β_1 be an ergodic action of a compact quantum group $\mathbb{G}_1$ on a unital C^*-algebra B. Denote by $\beta_1' : \mathcal{B} \to \mathcal{O}(\mathbb{G}_1) \odot \mathcal{B}$ the associated coaction of $\mathcal{O}(\mathbb{G}_1)$ on $\mathcal{B}$. Let X^x be as defined in* (2.5.1) *before proposition 2.5.1 and $T_{\beta_1'}$ as defined in definition 1.2.1. Then the following are equivalent:*

1. *there exists a unitary fiber functor ψ on $\mathbb{G}_1$ which induces a monoidal equivalence $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$ such that $\mathcal{B}$ is the ($\mathbb{G}_1$-$\mathbb{G}_2$)-bi-Galois object associated to ψ,*
2. *β_1 is of full quantum multiplicity,*
3. *the X^x are unitary for all irreducible representations x,*

4. *$T_{\beta_1'}$ is a Galois maps and hence $\mathcal{B}$ is a left $\mathcal{O}(\mathbb{G}_1)$-Galois object.*

In this spirit of this theorem, we can introduce the notion of isomorphism of unitary fiber functors, which is equivalent to the isomorphism of the associated Galois objects.

Definition 2.6.8 (Definition 3.10 in [27]). *Let ψ and ψ' be two unitary fiber functors on a compact quantum group $\mathbb{G}$. We say they are isomorphic if there exist unitaries $u_x \in B(\mathcal{H}_{\varphi(x)}, \mathcal{H}_{\psi(x)})$ such that*

$$\psi'(S) = (u_{y_1} \otimes \ldots \otimes u_{y_k})\psi(S)(u^*_{x_1} \otimes \ldots \otimes u^*_{x_r})$$

for all $S \in \mathrm{Mor}(y_1 \otimes \ldots \otimes y_k, x_1 \otimes \ldots \otimes x_r)$.

Proposition 2.6.9 ([27]). *Let ψ and ψ' be two unitary fiber functors on a compact quantum group $\mathbb{G}$. Let $\mathcal{B}_{\psi'}$ and $\mathcal{B}_\psi$ be the associated bi-Galois objects with respective coactions β_ψ, β'_ψ. Then ψ and ψ' are isomorphic as unitary fiber functors if and only if there exists a *-isomorphism $\lambda : \mathcal{B}_\psi \to \mathcal{B}_{\psi'}$ satisfying $(\lambda \otimes \mathrm{id})\beta_\psi = \beta_{\psi'}\lambda$.*

It is good to note again the equivalence with chapter 1. In chapter 1 we proved that if B is a left H-*-Galois object for a Hopf *-algebra H, then H and $\widetilde{{}^B H}$ have equivalent strict monoidal *-categories of *-comodules. Here we called the compact quantum groups monoidally equivalent if their representation categories are equivalent and proved later that there exists a Galois object. In chapter 1 it followed easily that there exists also a deformation of comodule-*-algebras. In the analytical context, this is proven by De Rijdt and Vander Vennet in [42].

Theorem 2.6.10 ([42]). *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two compact quantum groups and let $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$ be a monoidal equivalence between them. Let $\mathcal{B}, \beta_1, \beta_2, X^x$ be as in theorem 2.6.5. Suppose moreover that D_1 is a C^*-algebra with an action $\alpha_1 : D_1 \to D_1 \otimes C(\mathbb{G}_1)$ of $\mathbb{G}_1$ on D_1. Using the dense Hopf *-algebras, we have a coaction $\alpha_1 : \mathcal{D}_1 \to \mathcal{D}_1 \odot \mathcal{O}(\mathbb{G}_1)$ of $\mathcal{O}(\mathbb{G}_1)$ on $\mathcal{D}_1$ and we can define the *-algebra:*

$$\mathcal{D}_2 = \mathcal{D}_1 \underset{\mathcal{O}(\mathbb{G}_1)}{\boxdot} \mathcal{B} = \{a \in \mathcal{D}_1 \odot \mathcal{B} | (\alpha_1 \otimes \mathrm{id}_\mathcal{B})(a) = (\mathrm{id}_{\mathcal{D}_1} \otimes \beta_1)(a)\}.$$

Furthermore there exists a coaction $\alpha_2 = (\mathrm{id} \otimes \beta_2)_{|\mathcal{D}_2}$ of $\mathcal{O}(\mathbb{G}_2)$ on $\mathcal{D}_2$. If α_1 is ergodic, α_2 is ergodic as well and $\mathrm{mult}_q(x) = \mathrm{mult}_q(\varphi(x))$ for all $x \in \mathrm{Irred}(\mathbb{G}_1)$.

Moreover, in [42], the authors prove that the same construction with the inverse monoidal equivalence φ^{-1} will give $\mathcal{D}_1$ again up to isomorphism.

To end this section, we will show that via the inverse monoidal equivalence on the deformed quantum group, one can obtain the original data again. Let us first fix some notation.
Let $\mathbb{G}_1$ be a compact quantum group and ψ a unitary fiber functor on $\mathbb{G}_1$, inducing a monoidal equivalence $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$ with bi-Galois object $\mathcal{B}$. Denote by $\varphi^{-1} : \mathbb{G}_2 \to \mathbb{G}_1$ the inverse monoidal equivalence with bi-Galois object $\tilde{\mathcal{B}}$ generated by the matrix coefficients of unitaries $Z^y \in B(\mathcal{H}_{\varphi^{-1}(y)}, \mathcal{H}_y) \odot \tilde{\mathcal{B}}$, $y \in \mathrm{Irred}(\mathbb{G}_2)$ and coactions $\delta_1 : \tilde{B} \to \tilde{\mathcal{B}} \odot \mathcal{O}(\mathbb{G}_1)$ and $\delta_2 : \tilde{\mathcal{B}} \to \mathcal{O}(\mathbb{G}_2) \odot \tilde{\mathcal{B}}$ such that

$$(\mathrm{id} \otimes \delta_1) Z^y = Z^y_{12} U^{\varphi^{-1}(y)}_{13} \qquad \text{and} \qquad (\mathrm{id} \otimes \delta_2) Z^y = U^y_{12} Z^y_{13}.$$

Now we rephrase proposition 7.6 from [42] in our notations.

Proposition 2.6.11 ([42]). *Let $\mathbb{G}_1, \mathbb{G}_2, \varphi, \mathcal{B}$ and $\tilde{\mathcal{B}}$ as above. Then*

$$\pi : \mathcal{O}(\mathbb{G}_1) \to \mathcal{B} \underset{\mathcal{O}(\mathbb{G}_2)}{\boxdot} \tilde{\mathcal{B}} \qquad \textit{determined by} \qquad (\mathrm{id} \otimes \pi)(U^x) = X^x_{12} Z^{\varphi(x)}_{13}$$

is a *-isomorphism such that $(\mathrm{id}_{C(\mathbb{G}_1)} \otimes \pi)\Delta_1 = (\beta_1 \otimes \mathrm{id}_{\tilde{\mathcal{B}}})\pi$ and $(\pi \otimes \mathrm{id}_{C(\mathbb{G}_1)})\Delta_1 = (\mathrm{id}_{\mathcal{B}} \otimes \delta_1)\pi$.

Remark 2.6.12. *From the proof of proposition 7.6 from [42], one sees that the π here is in fact the same as the γ in the proof of theorem 2.5.4. From proposition 1.4.12 one finds that $\tilde{\mathcal{B}}$ is isomorphic with $\mathcal{B}^{op}$ as proved in proposition 3.2.7 of the thesis of An De Rijdt [41], but that the star structure should be altered to be compatible with the star-structure on $\mathcal{B}^{op}$ as defined in proposition 1.4.11. Denoting by $p_x : \mathcal{H}_x \to \mathcal{H}_{\overline{x}} : \xi \mapsto (\mathrm{id}_{\mathcal{H}_{\overline{x}}} \otimes \xi^*) t_{\overline{x}}$ and $P_x = p_x^* p_x$, we have*

$$\big((b^x_{ij})^{op}\big)^\star = \langle \xi^x_i, P_x \xi^x_k \rangle (b^x_{ks})^* \langle \xi^x_s, L_x \xi_j \rangle.$$

2.7 Conclusion

In this second chapter, we recalled the theory of compact quantum groups and emphasized on actions of CQG's of full quantum multiplicity. We proved that this is the analytical counterpart of coactions of Galois objects. Finally we described monoidal equivalences and observed that those are the analytical counterpart of the Hopf-Galois equivalence induced by Galois objects in the algebraic framework.

Chapter 3

Monoidal Deformations of spectral triples

In the first and second chapter we described a deformation method for Hopf algebras and for compact quantum groups, which are essentially equivalent (theorem 2.6.7). In this third chapter we will use this method to obtain a procedure to deform compact spectral triples, which are non-commutative generalizations of compact manifolds. The procedure developed here is new work, which is the heart and main result of [43].

The initial data are the following:

- a spectral triple (definition 3.1.1)
- a compact quantum group acting algebraically and by orientation preserving isometries on the spectral triple (definition 3.1.5)
- a unitary fiber functor on the compact quantum group (or a monoidal equivalence between the quantum group and another compact quantum group) (definitions 2.6.1 and 2.6.3).

This chapter is structured as follows. In the first section we remind some notions of non-commutative geometry. In the second, we develop the actual deformation procedure.

3.1 Spectral triples and compact quantum groups acting on them

Before we start with the description of the deformation procedure, we recapitulate the notion of spectral triples (which are non-commutative manifolds) and that of compact quantum groups acting on spectral triples.

Definition 3.1.1 ([32]). *A (compact) spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of*

1. *a unital *-algebra $\mathcal{A}$ acting as bounded operators on $\mathcal{H}$,*
2. *a Hilbert space $\mathcal{H}$,*
3. *an unbounded selfadjoint operator D on $\mathcal{H}$ with compact resolvent such that $\mathcal{A}\,\mathrm{dom}\,D \subset \mathrm{dom}\,D$ and $[D, a]$ is bounded for all $a \in \mathcal{A}$.*

The classical example comes from a classical manifold. If M is a compact spin manifold, let $C^\infty(M)$ be the algebra of smooth functions on M, S the spinor bundle, $L^2(M, S)$ the bundle of L^2 integrable spinors and D the classical Dirac operator. Then $(C^\infty(M), L^2(M, S), D)$ is a compact spectral triple.

There is even a reconstruction theorem ([32] and explained clearly in [54]): if $(\mathcal{A}, \mathcal{H}, D)$ is a compact spectral triple with commutative algebra $\mathcal{A}$ and satisfying some technical conditions, then the spectral triple is of the form $(C^\infty(M), L^2(M, S), D)$ for some compact spin manifold M.

Definition 3.1.2 ([32]). *Two spectral triples $(\mathcal{A}_1, \mathcal{H}_1, D_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2)$ are called isomorphic, if there exists an isomorphism of Hilbert spaces $\phi : \mathcal{H}_1 \to \mathcal{H}_2$ and an isomorphism of *-algebras $\lambda : \mathcal{A}_1 \to \mathcal{A}_2$ such that $\phi D_1 = D_2 \phi$ and $\phi(a\xi) = \lambda(a)\phi(\xi)$ for arbitrary $\xi \in \mathcal{H}_1, a \in \mathcal{A}_1$.*

In [15, 50] Bhowmick and Goswami described how compact quantum groups can act isometrically and orientation-preserving on a non-commutative manifold, i.e. a spectral triple.

Definition 3.1.3 ([15]). *Let $(\mathcal{A}, \mathcal{H}, D)$ be a compact spectral triple, $\mathbb{G} = (C(\mathbb{G}), \Delta)$ a compact quantum group and U a unitary representation of $\mathbb{G}$ on $\mathcal{H}$. Then $\mathbb{G}$ is said to act by orientation-preserving isometries on $(\mathcal{A}, \mathcal{H}, D)$ with U if*

- *for every state ϕ on $C(\mathbb{G})$, we have $U_\phi D = D U_\phi$ where $U_\phi := (\mathrm{id} \otimes \phi)(U)$,*

- $(\mathrm{id} \otimes \phi) \circ \alpha_U(a) \in \mathcal{A}''$ *for all* $a \in \mathcal{A}$ *and all states* ϕ *on* $C(\mathbb{G})$ *where* $\alpha_U(T) := U(T \otimes 1)U^*$ *for* $T \in B(\mathcal{H})$.

This definition is a very strong one: it ensures the existence of a universal object in the category of all compact quantum groups acting by orientation-preserving isometries. However, in some cases the second condition is too weak: the quantum group representation on $\mathcal{H}$ may behave badly with respect to the algebra $\mathcal{A}$ in the sense that the induced action of the CQG on $\mathcal{A}$ is not a CQG-action on the C^*-closure of $\mathcal{A}$. This is in some situations a disadvantage. Therefore, we note the following proposition of Goswami, found in [53].

Proposition 3.1.4. *Let* $(\mathcal{A}, \mathcal{H}, D)$, $(C(\mathbb{G}), \Delta)$ *and* U *be as above. Then there exists a unital* **-algebra* $\mathcal{A}_1$ *such that*

1. $\mathcal{A}_1$ *is SOT-dense in the von Neumann Algebra* $M = \mathcal{A}''$,
2. α_U *is algebraic on* $\mathcal{A}_1$, *i.e.* $(\alpha_U)_{|\mathcal{A}_1} : \mathcal{A}_1 \to \mathcal{A}_1 \odot \mathcal{O}(\mathbb{G})$,
3. $[D, a]$ *is bounded for every* $a \in \mathcal{A}_1$,
4. $(\mathcal{A}_1, \mathcal{H}, D)$ *is again a spectral triple.*

Proof. This follows from sections 4.4.3 and 4.4.4 and theorem 4.10 in [53]. □

Driven by proposition 3.1.4 we will use the following definition:

Definition 3.1.5. *Let* $(\mathcal{A}, \mathcal{H}, D)$ *be a compact spectral triple,* $\mathbb{G} = (C(\mathbb{G}), \Delta)$ *a compact quantum group and* U *a unitary representation of* $\mathbb{G}$ *on* $\mathcal{H}$. *Then* $\mathbb{G}$ *is said to act* ***algebraically*** *and by orientation-preserving isometries on* $(\mathcal{A}, \mathcal{H}, D)$ *with* U *if*

- *for every state* ϕ *on* $C(\mathbb{G})$, *we have* $U_\phi D = D U_\phi$ *where* $U_\phi := (\mathrm{id} \otimes \phi)(U)$,
- α_U *is algebraic on* $\mathcal{A}$, *i.e.* $(\alpha_U)_{|\mathcal{A}} : \mathcal{A} \to \mathcal{A} \odot \mathcal{O}(\mathbb{G})$ *where* $\alpha_U(T) := U(T \otimes 1)U^*$ *for* $T \in B(\mathcal{H})$.

In what follows, we will always work with compact quantum groups acting algebraically on $(\mathcal{A}, \mathcal{H}, D)$.

3.2 The box product for Hilbert space

In this section, we give some technical results about the analogue for Hilbert space of the box product $\mathcal{A} \boxdot \mathcal{B}$ introduced in the first chapter. Moreover, we describe the link between the algebraic box product and the box product for Hilbert spaces.

Definition 3.2.1. *Let $\mathbb{G}$ be a compact quantum group. Let U be a right unitary representation of $\mathbb{G}$ on a Hilbert space $\mathcal{H}_1$ and let V be a left unitary representation of $\mathbb{G}$ on a Hilbert space $\mathcal{H}_2$. Then we use the notation*

$$\mathcal{H}_1 \underset{C(\mathbb{G})}{\boxtimes} \mathcal{H}_2 := \{\xi \in \mathcal{H}_1 \otimes \mathcal{H}_2 \mid U_{12}\xi_{13} = V_{23}\xi_{13}\}$$

and call it the box product of $\mathcal{H}_1$ and $\mathcal{H}_2$.

Proposition 3.2.2. *Let $\mathbb{G}$ be a compact quantum group. Let U be a right unitary representation of $\mathbb{G}$ on a Hilbert space $\mathcal{H}_1$ and let V be a left unitary representation of $\mathbb{G}$ on a Hilbert space $\mathcal{H}_2$. Then $\mathcal{H}_1 \underset{C(\mathbb{G})}{\boxtimes} \mathcal{H}_2$ is a Hilbert space.*

Proof. It easy to see that $\mathcal{H}_1 \underset{C(\mathbb{G})}{\boxtimes} \mathcal{H}_2$ is a vector subspace of the tensor product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$. As the right resp. left representations u and v (i.e. the maps of proposition 2.2.4 associated to U and V) of $C(\mathbb{G})$ on $\mathcal{H}_1$ resp. $\mathcal{H}_2$ are continuous and $\mathcal{H}_1 \underset{C(\mathbb{G}_1)}{\boxtimes} \mathcal{H}_2$ is the kernel of $u \otimes \mathrm{id}_{\mathcal{H}_2} - \mathrm{id}_{\mathcal{H}_1} \otimes v$, $\mathcal{H}_1 \underset{C(\mathbb{G}_1)}{\boxtimes} \mathcal{H}_2$ is complete. □

Moreover, one can ask what the link is between the algebraic box product and this new box product for Hilbert spaces.

Proposition 3.2.3. *Let $\mathbb{G}$ be a compact quantum group and $\beta_0 : B \to C(\mathbb{G}) \otimes B$ an ergodic action of $\mathbb{G}$ on a unital C^*-algebra B. Denote by $\mathcal{B}$ the dense unital *-subalgebra of B with left coaction $\beta : \mathcal{B} \to \mathcal{O}(\mathbb{G}) \odot \mathcal{B}$. Denote moreover by ω the unique faithful state on $\mathcal{B}$ such that $(\mathrm{id}_{C(\mathbb{G})} \otimes \omega)\beta(b) = \omega(b)1_{C(\mathbb{G})}$.*

Defining $L^2(\mathcal{B})$ to be the GNS representation space of $\mathcal{B}$ with respect to ω and $\Lambda : \mathcal{B} \to L^2(\mathcal{B})$ the GNS map, we have:

1. *there exists a left unitary representation β' of $C(\mathbb{G})$ on $L^2(\mathcal{B})$ such that $\beta'(\Lambda(b)) = (\mathrm{id} \otimes \Lambda)(\beta(b))$ for all $b \in \mathcal{B}$.*
2. *β' is ergodic, i.e. if $\xi \in L^2(\mathcal{B})$ such that $\beta'(\xi) = 1 \otimes \xi$, then $\xi \in \mathbb{C}\Lambda(1_{\mathcal{B}})$.*

Proof. 1. For $\xi = \Lambda(b) \in \Lambda(\mathcal{B})$, let $\beta'(\xi) = (\mathrm{id} \otimes \Lambda)(\beta(b))$. Then β' is well defined on $\Lambda(\mathcal{B})$, as ω is faithful on $\mathcal{B}$ and hence Λ is injective. Now recall the $C(\mathbb{G}_1)$-valued inner product on the Hilbert module $C(\mathbb{G}_1) \otimes L^2(\mathcal{B})$ with $\langle a \otimes \xi, a' \otimes \xi' \rangle_{C(\mathbb{G}_1)} = \langle \xi, \xi' \rangle_{L^2(\mathcal{B})} a^* a'$. Then we have

$$\begin{aligned}\langle \beta'(\Lambda(b)), \beta'(\Lambda(b')) \rangle_{C(\mathbb{G}_1)} &= (\mathrm{id} \otimes \omega)(\beta(b^* b')) \\ &= \omega(b^* b') 1_{C(\mathbb{G}_1)} \\ &= \langle \Lambda(b), \Lambda(b') \rangle_{L^2(\mathcal{B})} 1_{C(\mathbb{G}_1)}\end{aligned}$$

where we used that ω is β-invariant. It proves that β' is indeed unitary in the sense of proposition 2.2.4(1) and hence can be extended to $L^2(\mathcal{B})$. Moreover

$$\begin{aligned}(\mathrm{id}_{C(\mathbb{G}_1)} \otimes \beta')\beta'(\Lambda(b)) &= (\mathrm{id}_{C(\mathbb{G}_1)} \otimes \beta')(\mathrm{id}_{C(\mathbb{G}_1)} \otimes \Lambda)\beta(b) \\ &= (\mathrm{id}_{C(\mathbb{G}_1)} \otimes \mathrm{id}_{C(\mathbb{G}_1)} \otimes \Lambda)(\mathrm{id}_{C(\mathbb{G}_1)} \otimes \beta)\beta(b) \\ &= (\mathrm{id}_{C(\mathbb{G}_1)} \otimes \mathrm{id}_{C(\mathbb{G}_1)} \otimes \Lambda)(\Delta \otimes \mathrm{id}_{\mathcal{B}})\beta(b) \\ &= (\Delta \otimes \mathrm{id}_{L^2(\mathcal{B})})\beta'(\Lambda(b))\end{aligned}$$

which extends by continuity to arbitrary $\xi \in L^2(\mathcal{B})$. This proves proposition 2.2.4(2). Finally, by the density condition in definition 2.4.1(2), we obtain proposition 2.2.4(3). This proves the statement.

2. Let $\xi \in L^2(\mathcal{B})$ and suppose that $\beta'(\xi) = 1 \otimes \xi$. Then note that

$$\xi = (h \otimes \mathrm{id}_{L^2(\mathcal{B})})(1_{C(\mathbb{G}_1)} \otimes \xi) = (h \otimes \mathrm{id}_{L^2(\mathcal{B})})(\beta'(\xi))$$

and that

$$(h \otimes \mathrm{id}_{L^2(\mathcal{B})})(\beta'(\Lambda(b))) = (h \otimes \Lambda)\beta(b) = \Lambda(\omega(b) 1_{\mathcal{B}})$$

for arbitrary $b \in B$ and hence

$$\|(h \otimes \mathrm{id}_{L^2(\mathcal{B})})\beta'(\Lambda(b))\|_{L^2(\mathcal{B})} = |\omega(b)|^2 \leq \omega(b^* b)$$

which makes $(h \otimes \mathrm{id}_{L^2(\mathcal{B})})\beta'$ a contractive and hence continuous map on $\Lambda(\mathcal{B})$. Extending it continuously, we obtain an element P of $B(L^2(\mathcal{B}))$. Hence, taking a sequence $(b_n)_n$ in $\mathcal{B}$ such that $\Lambda(b_n) \to \xi$ in L^2-norm, we have $P(\Lambda(b_n)) \to P(\xi) = \xi$. Moreover, we have

$$P(\Lambda(b_n)) = (h \otimes \mathrm{id}_{L^2(\mathcal{B})})(\beta'(\Lambda(b_n))) = (h \otimes \Lambda)\beta(b_n) = \Lambda(\omega(b_n) 1_{\mathcal{B}}) \in \mathbb{C}\Lambda(1_{\mathcal{B}})$$

for every $n \in \mathbb{N}$. Hence we may conclude that $\xi \in \mathbb{C}\Lambda(1_{\mathcal{B}})$.

$\square$

Furthermore, if we have two algebra's with compatible coaction, one can make the construction $L^2(\mathcal{B}_2) \underset{C(\mathbb{G})}{\boxtimes} L^2(\mathcal{B}_1)$. It turns out that this is isomorphic with $L^2(\mathcal{B}_2 \underset{\mathcal{O}(\mathbb{G})}{\boxdot} \mathcal{B}_1)$.

Proposition 3.2.4. *Let $\mathbb{G}$ be a compact quantum group, $\mathcal{B}_1$ and $\mathcal{B}_2$ be unital *-algebras with ergodic coactions*

$$\beta_1 : \mathcal{B}_1 \to \mathcal{O}(\mathbb{G}) \odot \mathcal{B}_1 \qquad \text{and} \qquad \beta_2 : \mathcal{B}_2 \to \mathcal{B}_2 \odot \mathcal{O}(\mathbb{G}).$$

Moreover suppose $\mathcal{B}_1$ resp. $\mathcal{B}_2$ are equipped with faithful states ω_1 resp. ω_2 such that $(\mathrm{id}_{\mathcal{O}(\mathbb{G})} \otimes \omega_1)\beta_1(b) = \omega_1(b)1_{\mathcal{O}(\mathbb{G})}$ and $(\omega_2 \otimes \mathrm{id}_{\mathcal{O}(\mathbb{G})})\beta_2(b) = \omega_2(b)1_{\mathcal{O}(\mathbb{G})}$ and such that β_1 and β_2 extend to ergodic actions $\overline{\beta_1} : B_{1r} \to C(\mathbb{G}) \otimes B_{1r}$ and $\overline{\beta_2} : B_{2r} \to B_{2r} \otimes C(\mathbb{G})$ of $\mathbb{G}$ on B_{1r} resp. B_{2r}.

Then $L^2(\mathcal{B}_2 \underset{\mathcal{O}(\mathbb{G})}{\boxdot} \mathcal{B}_1) \cong L^2(\mathcal{B}_2) \underset{C(\mathbb{G})}{\boxtimes} L^2(\mathcal{B}_1)$.

Proof. Let Λ_1 be the GNS-map of $\mathcal{B}_1$ with respect to ω_1, Λ_2 be the GNS-map of $\mathcal{B}_2$ with respect to ω_2 and Λ the GNS-map of $\mathcal{B}_2 \underset{\mathcal{O}(\mathbb{G})}{\boxdot} \mathcal{B}_1$ with respect to $\omega_2 \otimes \omega_1$. Then it is easy to see that for $x \in \mathrm{Irred}(\mathbb{G})$, the map

$$\phi_x : \Lambda\big((\mathcal{B}_2)_x \underset{\mathcal{O}(\mathbb{G})}{\boxdot} (\mathcal{B}_1)_x\big) \to \Lambda_2((\mathcal{B}_2)_x) \underset{C(\mathbb{G})}{\boxtimes} \Lambda_1((\mathcal{B}_1)_x) : \Lambda(z) \mapsto (\Lambda_1 \otimes \Lambda_2)(z)$$

is an isomorphism of (finite dimensional) Hilbert spaces. Hence we have

$$\begin{aligned}
L^2(\mathcal{B}_2 \underset{\mathcal{O}(\mathbb{G})}{\boxdot} \mathcal{B}_1) &\cong L^2\big(\oplus_{x\in\mathrm{Irred}(\mathbb{G})} (\mathcal{B}_2)_x \underset{\mathcal{O}(\mathbb{G})}{\boxdot} \oplus_{x\in\mathrm{Irred}(\mathbb{G})}(\mathcal{B}_1)_x\big) \\
&\cong \oplus^{L^2}_{x\in\mathrm{Irred}(\mathbb{G})} L^2\big((\mathcal{B}_2)_x \underset{\mathcal{O}(\mathbb{G})}{\boxdot} (\mathcal{B}_1)_x\big) \\
&\cong \oplus^{L^2}_{x\in\mathrm{Irred}(\mathbb{G})} \Lambda\big((\mathcal{B}_2)_x \underset{\mathcal{O}(\mathbb{G})}{\boxdot} (\mathcal{B}_1)_x\big) \\
&\cong \oplus^{L^2}_{x\in\mathrm{Irred}(\mathbb{G})} \Big(\Lambda_2((\mathcal{B}_2)_x) \underset{C(\mathbb{G})}{\boxtimes} \Lambda_1((\mathcal{B}_1)_x)\Big) \\
&\cong \oplus^{L^2}_{x\in\mathrm{Irred}(\mathbb{G})} \Big(L^2((\mathcal{B}_2)_x) \underset{C(\mathbb{G})}{\boxtimes} L^2((\mathcal{B}_1)_x)\Big) \\
&\cong \oplus^{L^2}_{x\in\mathrm{Irred}(\mathbb{G})} L^2((\mathcal{B}_2)_x) \underset{C(\mathbb{G})}{\boxtimes} \oplus^{L^2}_{x\in\mathrm{Irred}(\mathbb{G})} L^2((\mathcal{B}_1)_x) \\
&\cong L^2(\mathcal{B}_2) \underset{C(\mathbb{G})}{\boxtimes} L^2(\mathcal{B}_1)
\end{aligned}$$

where by $\oplus^{L^2}_{x\in\mathrm{Irred}(\mathbb{G})}$ we mean the L^2-direct sum. This completes the proof.

□

3.3 Deformation procedure for spectral triples

In this section we describe the actual deformation procedure for spectral triples step by step. The deformation data to start with are:

- a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ of compact type,
- a compact quantum group $\mathbb{G}_1 = (C(\mathbb{G}_1), \Delta_1)$ acting algebraically and by orientation-preserving isometries on $(\mathcal{A}, \mathcal{H}, D)$ with a unitary representation U and
- a unitary fiber functor ψ on $\mathbb{G}_1$.

The unitary fiber functor will induce a new compact quantum group $\mathbb{G}_2$ and a *-algebra $\mathcal{B}$ with left resp. right coaction of $\mathcal{O}(\mathbb{G}_1)$ resp. $\mathcal{O}(\mathbb{G}_2)$ on $\mathcal{B}$ as described in theorem 2.6.5. Using this, one can deform the data one by one to obtain a new, deformed, spectral triple on which $\mathbb{G}_2$ acts algebraically and by orientation-preserving isometries.
To be more precise, consider the following:

1. As ψ is a unitary fiber functor on $\mathbb{G}_1$, following theorem 2.6.4 there exists a compact quantum group $\mathbb{G}_2$ and a monoidal equivalence $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$. We will call $\mathbb{G}_2$ the deformed quantum group.
2. Let $(\mathcal{B}, \omega)$ be the *-algebra and faithful invariant state associated to φ with the coactions
$$\beta_1 : \mathcal{B} \to \mathcal{O}(\mathbb{G}_1) \odot \mathcal{B} \qquad \text{and} \qquad \beta_2 : \mathcal{B} \to \mathcal{B} \odot \mathcal{O}(\mathbb{G}_2)$$
from theorem 2.6.5.
3. Let $X^x \in B(\mathcal{H}_{\varphi(x)}, \mathcal{H}_x) \odot \mathcal{B}$ be the unitaries such that
$$(\mathrm{id} \otimes \beta_1)X^x = U^x_{12}X^x_{13} \qquad \text{and} \qquad (\mathrm{id} \otimes \beta_2)X^x = X^x_{12}U^{\varphi(x)}_{13}.$$
4. Let $u : \mathcal{H} \to \mathcal{H} \otimes C(\mathbb{G}_1) : \xi \mapsto U(\xi \otimes 1)$ be the representation of $\mathbb{G}_1$ on $\mathcal{H}$ and denote by $\alpha = \mathrm{ad}_U : \mathcal{A} \to \mathcal{A} \odot \mathcal{O}(\mathbb{G}_1) : a \to U(a \otimes 1_{C(\mathbb{G}_1)})U^*$ the algebraic coaction of $\mathbb{G}_1$ on $\mathcal{A}$.

Definition 3.3.1. *With the data from above, the Hilbert space*

$$\mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}) = \{\xi \in \mathcal{H} \otimes L^2(\mathcal{B}) | U_{12}\xi_{13} = (\mathrm{id}_{\mathcal{H}} \otimes \beta_1')(\xi)\}$$

with β_1' as in proposition 3.2.3 is called the deformed Hilbert space $\tilde{\mathcal{H}}$.

Proposition 3.3.2. *We have*

1. $\mathcal{H}_x \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})$ *is isomorphic with* $\mathcal{H}_{\varphi(x)}$ *for all* $x \in \mathrm{Irred}(\mathbb{G}_1)$.
2. $$\mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}) = \bigoplus_{\lambda \in \sigma(D)} V_\lambda \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})$$

   where V_λ is the eigenspace of $\lambda \in \sigma(D)$.
3. $V_\lambda \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})$ *is finite dimensional for each* $\lambda \in \sigma(D)$.

Motivated by the first fact, we will call $\mathcal{H}_{\varphi(x)}$ the deformation of $\mathcal{H}_x$ for $x \in \mathrm{Irred}(\mathbb{G}_1)$.

Proof. 1. Note that, for $x \in \mathrm{Irred}(\mathbb{G}_1)$ and $\xi \in \mathcal{H}_{\varphi(x)}$, one has

$$\begin{aligned}
&(\mathrm{id}_{\mathcal{H}_x} \otimes \beta_1')(X^x(\xi \otimes \Lambda(1_{\mathcal{B}}))) \\
&= (\mathrm{id}_{\mathcal{H}_x} \otimes (\mathrm{id}_{C(\mathbb{G}_1)} \otimes \Lambda)\beta_1) X^x(\xi \otimes 1_{\mathcal{B}}) \\
&= (\mathrm{id}_{\mathcal{H}_x} \otimes \mathrm{id}_{C(\mathbb{G}_1)} \otimes \Lambda) U^x_{12} X^x_{13} (\xi \otimes 1_{C(\mathbb{G}_1)} \otimes 1_{\mathcal{B}}) \\
&= U^x_{12} \big(X^x(\xi \otimes \Lambda(1_{\mathcal{B}}))\big)_{13}
\end{aligned}$$

proving $X^x(\xi \otimes \Lambda(1_{\mathcal{B}})) \in \mathcal{H}_x \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})$. Also note that $(\mathrm{id}_{\mathcal{H}_{\varphi(x)}} \otimes \omega_1')(X^{x*} z) \in \mathcal{H}_{\varphi(x)}$ for $z \in \mathcal{H}_x \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})$, where $\omega_1' : L^2(\mathcal{B}) \to \mathbb{C} : \eta \mapsto \langle \Lambda(1_{\mathcal{B}}), \eta \rangle$.

Hence we can define the following maps:

$$f_x : \mathcal{H}_{\varphi(x)} \to \mathcal{H}_x \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}) : \xi \mapsto X^x(\xi \otimes \Lambda(1_{\mathcal{B}}))$$

$$g_x : \mathcal{H}_x \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}) \to \mathcal{H}_{\varphi(x)} : z \mapsto (\mathrm{id}_{\mathcal{H}_{\varphi(x)}} \otimes \omega_1')(X^{x*} z).$$

Now, one can verify that

$$(\mathrm{id} \otimes \beta_1')(X^{x*} z) = (\mathrm{id} \otimes \beta_1)(X^{x*})(\mathrm{id} \otimes \beta_1')(z) = X^x_{13}{}^* U^x_{12}{}^* U^x_{12} z_{13} = (X^{x*} z)_{13}$$

and as β_1' is an ergodic representation, we have $X^{x*}z \in \mathcal{H}_{\varphi(x)} \otimes \mathbb{C}\Lambda(1_{\mathcal{B}})$. Hence $(\mathrm{id} \otimes \omega_1')(X^{x*}z) \otimes \Lambda(1_{\mathcal{B}}) = X^{x*}z$. One can check now that indeed f_x and g_x are inverse to each other:

$$f_x(g_x(z)) = X^x(g_x(z) \otimes \Lambda(1_{\mathcal{B}})) = X^x X^{x*} z = z$$

for $z \in \mathcal{H}_x \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})$ and

$$g_x(f_x(\xi)) = (\mathrm{id}_{\mathcal{H}_{\varphi(x)}} \otimes \omega_1')(X^{x*} f_x(\xi)) = (\mathrm{id}_{\mathcal{H}_{\varphi(x)}} \otimes \omega_1')\big(X^{x*}X^x(\xi \otimes \Lambda(1_{\mathcal{B}}))\big) = \xi$$

for $\xi \in \mathcal{H}_{\varphi(x)}$ which proves f_x and g_x are inverse maps. Finally, using that X^x is unitary, it is easy to see that f_x and g_x are also unitary.

2. Note first that as D has compact resolvent, there exist a sequence of real eigenvalues $(\lambda_n)_n$ with finite dimensional eigenspaces and such that $\lim_{n\to\infty} \lambda_n = \infty$ (see appendix, proposition 3.4.3). Hence we have $\mathcal{H} = \bigoplus_{\lambda \in \sigma(D)} V_\lambda$ and also $\mathcal{H} \otimes L^2(\mathcal{B}) = \bigoplus_{\lambda\in\sigma(D)} V_\lambda \otimes L^2(\mathcal{B})$. As U and D commute, there is a subrepresentation U_λ of U on V_λ for every eigenvalue λ such that with $V_\lambda \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}) := \{\xi \in V_\lambda \otimes L^2(\mathcal{B}) | (U_\lambda)_{12}\xi_{13} = (\mathrm{id} \otimes \beta_1')\xi\}$ we have
$$\mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}) = \bigoplus_{\lambda\in\sigma(D)} V_\lambda \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}).$$

3. Finally, decomposing U_λ into irreducible representations of $\mathbb{G}_1$, we have $V_\lambda = \mathcal{H}_{x_1} \oplus \ldots \oplus \mathcal{H}_{x_l}$ for some $l \in \mathbb{N}$, $x_i \in \mathrm{Irred}(\mathbb{G}_1)$. Hence
$$\begin{aligned} V_\lambda \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}) &= (\mathcal{H}_{x_1} \oplus \ldots \oplus \mathcal{H}_{x_l}) \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}) \\ &= (\mathcal{H}_{x_1} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})) \oplus \ldots \oplus (\mathcal{H}_{x_l} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})) \\ &= \mathcal{H}_{\varphi(x_1)} \oplus \ldots \oplus \mathcal{H}_{\varphi(x_l)} \end{aligned} \tag{3.3.1}$$
where we used the first statement of this proposition in the last equality. This last direct sum of finite dimensional Hilbert spaces assures $V_\lambda \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})$ to be finite dimensional.

□

Remark 3.3.3. *Given a monoidal equivalence $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$ and a unitary representation U of $\mathbb{G}_1$ on a Hilbert space $\mathcal{H}$, the intuitive way to deform $\mathcal{H}$ is to decompose it into $\mathcal{H} = \oplus_{x\in\mathrm{Irred}(\mathbb{G}_1)}\mathcal{H}_x$ and defining $\tilde{\mathcal{H}} = \oplus_{x\in\mathrm{Irred}(\mathbb{G}_1)}\mathcal{H}_{\varphi(x)}$. Proposition 3.3.2(1) proves this is equivalent with our method, as $\tilde{\mathcal{H}} = \mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})$.*

Proposition 3.3.4. *$D \otimes \mathrm{id}_{L^2(\mathcal{B})}$ restricts to an unbounded selfadjoint operator $\tilde{D}$ on $\tilde{\mathcal{H}} = \mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})$. Moreover, $\tilde{D}$ has compact resolvent with eigenspaces $V_\lambda \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})$.*

Note that $D \otimes \mathrm{id}_{L^2(\mathcal{B})}$ is a well defined operator by proposition 3.4.6 in the appendix.

Proof. As D has compact resolvent, its restriction D_λ to the eigenspace V_λ is multiplication with λ for every λ in the spectrum. Therefore $D_\lambda \otimes \mathrm{id}$ can be restricted to $V_\lambda \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}) \subset V_\lambda \otimes L^2(\mathcal{B})$. As $\mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}) = \oplus_{\lambda \in \sigma(D)} V_\lambda \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})$ (proposition 3.3.2), we can take the direct sum to get an unbounded operator $\tilde{D}$ on $\mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})$ with domain $\{\xi \in \bigoplus_{\lambda \in \sigma(D)} V_\lambda \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}) | \sum_{\lambda \in \sigma(D)} |\lambda|^2 \|P_\lambda(\xi)\|^2 < \infty\}$, where P_λ is the projection $\mathcal{H} \to V_\lambda$. Now note that the following facts hold by construction:

- $\sigma(\tilde{D}) = \sigma(D) \subset \mathbb{R}$ and $V_\lambda \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})$ is the eigenspace corresponding to λ hence $\tilde{D}$ has compact resolvent by propositions 3.3.2(3) and 3.4.3;
- $\tilde{D} = \sum_{\lambda \in \sigma(D)} \lambda (P_\lambda \otimes \mathrm{id})$,
- $\mathrm{dom}(\tilde{D}) = \mathrm{dom}(D \otimes \mathrm{id}_{L^2(\mathcal{B})}) \cap \mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})$.

Hence $\tilde{D}$ is the restriction of $D \otimes \mathrm{id}_{L^2(\mathcal{B})}$ to $\mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})$. Moreover, $D \otimes \mathrm{id}_{L^2(\mathcal{B})}$ is selfadjoint by proposition 3.4.6 in the appendix and hence $\tilde{D}$ is symmetric. As it densely defined by construction, it suffices to prove that it is closed by proposition 3.4.7. Now note that if $\zeta_i \in \mathrm{dom}(\tilde{D})$ converges to $\zeta \in \mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})$ and $\tilde{D}(\zeta_i) \to \eta \in \mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})$, then $D \otimes \mathrm{id}_{L^2(\mathcal{B})}(\zeta) = \tilde{D}\zeta_i \to \eta$ and hence $\eta \in \mathrm{dom}(D \otimes \mathrm{id}_{L^2(\mathcal{B})}) \cap \mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}) = \mathrm{dom}(\tilde{D})$ as $D \otimes \mathrm{id}_{L^2(\mathcal{B})}$ is closed. Furthermore $D \otimes \mathrm{id}_{L^2(\mathcal{B})}(\zeta) = \tilde{D}\zeta = \eta$ and hence $\tilde{D}$ is closed. This completes the proof.

□

Proposition 3.3.5. *Define $\tilde{\mathcal{A}} = \mathcal{A} \underset{\mathcal{O}(\mathbb{G}_1)}{\boxdot} \mathcal{B} := \{z \in \mathcal{A} \odot \mathcal{B} \,|(\alpha \otimes \mathrm{id}_{\mathcal{B}})(z) = (\mathrm{id}_{\mathcal{A}} \otimes \beta_1)(z)\}$. Then $\tilde{\mathcal{A}}$ is a *-algebra endowed with a coaction $\alpha_2 = (\mathrm{id} \otimes \beta_2)_{|_{\tilde{\mathcal{A}}}} : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}} \odot \mathcal{O}(\mathbb{G}_2)$ of $\mathcal{O}(\mathbb{G}_2)$. Moreover, $\tilde{\mathcal{A}}$ acts by bounded operators on $\tilde{\mathcal{H}}$: for*

$z \in \tilde{\mathcal{A}}$, we have $\tilde{L}_z : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}} : \zeta \mapsto z\zeta$ such that $\tilde{L}_z\zeta = \sum_{i,j} a_i\xi_j \otimes b_i\eta_j$ if $z = \sum_i a_i \otimes b_i$ and $\zeta = \sum_j \xi_j \otimes \eta_j \in \mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})$.

Proof. The first statement is an application of theorem 2.6.10. For the second, note that $\tilde{\mathcal{A}} \subset \mathcal{A} \odot \mathcal{B}$ and $\mathcal{A} \odot \mathcal{B}$ acts by bounded operators on $\mathcal{H} \otimes L^2(\mathcal{B})$. Hence it suffices to prove that $\tilde{\mathcal{A}}$ leaves $\tilde{\mathcal{H}}$ invariant. Indeed, we have for $a \in \tilde{\mathcal{A}}, \xi \in \tilde{\mathcal{H}}$

$$\begin{aligned}
(\mathrm{id}_{\mathcal{H}} \otimes \beta_1')(a\xi) &= (\mathrm{id}_{\mathcal{A}} \otimes \beta_1)(a)(\mathrm{id}_{\mathcal{H}} \otimes \beta_1')(\xi) \\
&= (\alpha \otimes \mathrm{id}_{\mathcal{B}})(a)U_{12}\xi_{13} \\
&= U_{12}a_{13}U_{12}^*U_{12}\xi_{13} \\
&= U_{12}(a\xi)_{13}.
\end{aligned}$$

This concludes the proof. □

Theorem 3.3.6. *$(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$ constitutes a spectral triple.*

Proof. Combining all the previous propositions, it suffices to prove that the commutator of $\tilde{D}$ with an element $a \in \tilde{\mathcal{A}}$ is bounded. For that, we will first prove that $\tilde{\mathcal{A}}$ leaves the domain of $\tilde{D}$ invariant and secondly we will proof that the commutator of $\tilde{D}$ with an arbitrary $a \in \tilde{\mathcal{A}}$ is bounded. Let z be an arbitrary element in $\mathcal{A} \odot \mathcal{B}$ and let ξ be an arbitrary nonzero vector in $\mathrm{dom}(D \otimes \mathrm{id})$. We will prove $z\xi \in \mathrm{dom}(D \otimes \mathrm{id})$. As $\xi \in \mathrm{dom}(D \otimes \mathrm{id})$, there exists a sequence ξ_n in $\mathrm{dom}(D) \odot L^2(\mathcal{B})$ such that simultaneously $\xi_n \to \xi$ and $(D \otimes \mathrm{id})\xi_n \to (D \otimes \mathrm{id})\xi$ for $n \to \infty$. Now we note three facts:

- as $\mathcal{A}$ leaves the domain of D invariant, $\mathcal{A} \odot \mathcal{B}$ leaves the core $\mathrm{dom}(D) \odot L^2(\mathcal{B})$ of $D \otimes \mathrm{id}$ invariant and hence $z\xi_n \in \mathrm{dom}(D) \odot L^2(\mathcal{B})$ for all n.
- Moreover, writing $z = \sum_{i=1}^m a_i \otimes b_i$ with $m \in \mathbb{N}$, one has

  $$[D \otimes \mathrm{id}, z] = [D \otimes \mathrm{id}, \sum_{i=1}^m a_i \otimes b_i] = \sum_{i=1}^m [D, a_i] \otimes b_i$$

  which is bounded on $\mathrm{dom}(D) \odot L^2(\mathcal{B})$ as $\mathcal{A}$ has bounded commutator with D.
- Furthermore, as $(D \otimes \mathrm{id})\xi_n \to (D \otimes \mathrm{id})\xi$, $(D \otimes \mathrm{id})\xi_n$ is a Cauchy sequence.

Combining the three, one has

$$\begin{aligned}
& \big\|(D\otimes \mathrm{id})z(\xi_n)-(D\otimes \mathrm{id})z(\xi_k)\big\| \\
= \;& \big\|z(D\otimes \mathrm{id})(\xi_n-\xi_k)+\big((D\otimes \mathrm{id})z-z(D\otimes \mathrm{id})(\xi_n-\xi_k)\big)\big\| \\
\leq \;& \|z(D\otimes \mathrm{id})(\xi_n-\xi_k)\|+\|[D\otimes \mathrm{id},z]\|_{B(\mathrm{dom}(D)\odot L^2(\mathcal{B}))}\|(\xi_n-\xi_k)\|
\end{aligned}$$

which converges to 0 as $z\in\mathcal{A}\odot\mathcal{B}$ is a bounded operator on $\mathcal{H}\otimes L^2(\mathcal{B})$. Hence $(D\otimes \mathrm{id})z(\xi_n)_n$ is a Cauchy sequence and thus converging in $\mathcal{H}\otimes L^2(\mathcal{B})$. As the $z\xi_n$ are elements of the core converging to $z\xi$ and $((D\otimes \mathrm{id})z(\xi_n))_n$ converges, we know that $z\xi\in\mathrm{dom}(D\otimes \mathrm{id})$ and $(D\otimes \mathrm{id})z\xi_n\to(D\otimes \mathrm{id})z\xi$. We can conclude that $(\mathcal{A}\odot\mathcal{B})(\mathrm{dom}(D\otimes \mathrm{id}))\subset\mathrm{dom}(D\otimes \mathrm{id})$.

Now note that

- $\mathrm{dom}\,\tilde{D}=\{\xi\in\mathcal{H}\underset{C(\mathbb{G}_1)}{\boxtimes}L^2(\mathcal{B})\mid\sum_{\lambda\in\sigma(D)}|\lambda|^2\|\tilde{P}_\lambda\xi\|^2\}=\mathrm{dom}(D\otimes \mathrm{id})\cap\mathcal{H}\underset{C(\mathbb{G}_1)}{\boxtimes}L^2(\mathcal{B})$,
- as $\tilde{\mathcal{A}}\subset\mathcal{A}\odot\mathcal{B}$, $\tilde{\mathcal{A}}(\mathrm{dom}(D\otimes \mathrm{id}))\subset\mathrm{dom}(D\otimes \mathrm{id})$ and
- $\tilde{\mathcal{A}}(\mathcal{H}\underset{C(\mathbb{G}_1)}{\boxtimes}L^2(\mathcal{B}))\subset\mathcal{H}\underset{C(\mathbb{G}_1)}{\boxtimes}L^2(\mathcal{B})$ (proposition 3.3.5).

Then it follows directly that $\tilde{\mathcal{A}}(\mathrm{dom}(\tilde{D}))\subset\mathrm{dom}(\tilde{D})$.

Finally, we prove that $\tilde{D}z-z\tilde{D}$ is indeed bounded on the domain of $\tilde{D}$. Let ξ be an arbitrary element of $\mathrm{dom}(\tilde{D})$ and take a sequence $(\xi_n)_n$ in $\mathrm{dom}(D)\odot L^2(\mathcal{B})$ converging to ξ. Then we know from above, that simultaneously

$$\begin{aligned}
\xi_n &\to \xi, \\
(D\otimes \mathrm{id})z\xi_n &\to (D\otimes \mathrm{id})z\xi, \\
z(D\otimes \mathrm{id})\xi_n &\to z(D\otimes \mathrm{id})\xi
\end{aligned}$$

and that $[D\otimes \mathrm{id},z]$ is bounded on $\mathrm{dom}(D)\odot L^2(\mathcal{B})$ (let's say $\|[D\otimes \mathrm{id},z]\|_{B(\mathrm{dom}(D)\odot L^2(\mathcal{B}))}=M$). Combining that, one can take $n\in\mathbb{N}$ large enough such that $\|\xi_n\|\leq 2\|\xi\|$ and $\|(D\otimes \mathrm{id})z\xi_n-z(D\otimes \mathrm{id})\xi_n\|\geq\|(D\otimes \mathrm{id})z\xi-z(D\otimes$

id)$\xi\| - \|\xi\|$, and hence

$$\begin{aligned}\|\tilde{D}z\xi - z\tilde{D}\xi\| &= \|(D\otimes \mathrm{id})z\xi - z(D\otimes \mathrm{id})\xi\|\\ &\leq \|(D\otimes \mathrm{id})z\xi_n - z(D\otimes \mathrm{id})\xi_n\| + \|\xi\|\\ &\leq M\|\xi_n\| + \|\xi\|\\ &\leq (2M+1)\|\xi\|\end{aligned}$$

proving $\tilde{D}z - z\tilde{D}$ to be indeed bounded on the domain $\mathrm{dom}(\tilde{D})$.

□

With the previous theorem, the deformation $(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$ of $(\mathcal{A}, \mathcal{H}, D)$ is well defined. The next step is to prove that $\mathbb{G}_2$ acts algebraically and by orientation preserving isometries on $(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$.

Theorem 3.3.7. *There exists a unitary representation $\tilde{U}$ of $C(\mathbb{G}_2)$ on $\mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})$ such that $\mathbb{G}_2$ acts algebraically and by orientation-preserving isometries on $(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$ with $\tilde{U}$.*

Proof. Using the coaction $\beta_2 : \mathcal{B} \to \mathcal{B} \odot \mathcal{O}(\mathbb{G}_2)$, one can construct, along the lines of Lemma 5 in [28] and the discussion above it, a unitary $\tilde{U}_0 : L^2(\mathcal{B}) \otimes C(\mathbb{G}_2) \to L^2(\mathcal{B}) \otimes C(\mathbb{G}_2)$ by

$$\tilde{U}_0\big(\sum_i \Lambda(b_i) \otimes a_i\big) = \sum_i (\Lambda \otimes \mathrm{id}_{C(\mathbb{G}_2)})\big(\beta_2(b_i)(1_{\mathcal{B}} \otimes a_i)\big).$$

Moreover, we know this is a unitary representation $\tilde{U}_0 \in \mathcal{M}(\mathcal{K}(L^2(\mathcal{B})) \otimes C(\mathbb{G}_2))$ and furthermore,

$$\beta_2(b) = \tilde{U}_0(b \otimes \mathrm{id}_{C(\mathbb{G}_2)})\tilde{U}_0^*. \tag{3.3.2}$$

Now one can prove that $\mathrm{id}_{\mathcal{H}} \otimes \tilde{U}_0 \in \mathcal{M}(\mathcal{K}(\mathcal{H} \otimes L^2(\mathcal{B})) \otimes C(\mathbb{G}_2))$ restricts to a representation $\tilde{U} \in \mathcal{M}(\mathcal{K}(\mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})) \otimes C(\mathbb{G}_2))$. First, note that as β_1 and β_2

commute, one has for $b_i \in \mathcal{B}$, $a_i \in C(\mathbb{G}_2)$,

$$\begin{aligned}
&(\beta_1' \otimes \mathrm{id}_{C(\mathbb{G}_2)})\tilde{U}_0(\sum_i \Lambda(b_i) \otimes a_i) \\
&= (\beta_1' \otimes \mathrm{id}_{C(\mathbb{G}_2)})\Big(\sum_i (\Lambda \otimes \mathrm{id}_{C(\mathbb{G}_2)})\big(\beta_2(b_i)(1_{\mathcal{B}} \otimes a_i)\big)\Big) \\
&= \sum_i \big((\mathrm{id}_{C(\mathbb{G}_1)} \otimes \Lambda)\beta_1 \otimes \mathrm{id}_{C(\mathbb{G}_2)}\big)\big(\beta_2(b_i)(1_{\mathcal{B}} \otimes a_i)\big) \\
&= \sum_i \big(\mathrm{id}_{C(\mathbb{G}_1)} \otimes (\Lambda \otimes \mathrm{id}_{C(\mathbb{G}_2)})\beta_2\big)\big(\beta_1(b_i)\big)(1_{C(\mathbb{G}_1)} \otimes 1_{\mathcal{B}} \otimes a_i) \\
&= \sum_i (\mathrm{id}_{C(\mathbb{G}_1)} \otimes \tilde{U}_0)\big((\mathrm{id}_{C(\mathbb{G}_1)} \otimes \Lambda)\beta_1(b_i) \otimes a_i\big) \\
&= (\mathrm{id}_{C(\mathbb{G}_1)} \otimes \tilde{U}_0)(\beta_1' \otimes \mathrm{id}_{C(\mathbb{G}_2)})(\sum_i \Lambda(b_i) \otimes a_i)
\end{aligned}$$

and hence $(\beta_1' \otimes \mathrm{id}_{C(\mathbb{G}_2)})\tilde{U}_0 = (\mathrm{id}_{C(\mathbb{G}_1)} \otimes \tilde{U}_0)(\beta_1' \otimes \mathrm{id}_{C(\mathbb{G}_2)})$.

Moreover for $\xi \in \mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})$ and $a \in C(\mathbb{G}_2)$ one has

$$\begin{aligned}
&(\mathrm{id}_{\mathcal{H}} \otimes \beta_1' \otimes \mathrm{id}_{C(\mathbb{G}_2)})(\mathrm{id}_{\mathcal{H}} \otimes \tilde{U}_0)(\xi \otimes a) \\
&= (\mathrm{id}_{\mathcal{H}} \otimes \mathrm{id}_{C(\mathbb{G}_1)} \otimes \tilde{U}_0)(\mathrm{id}_{\mathcal{H}} \otimes \beta_1' \otimes \mathrm{id}_{C(\mathbb{G}_2)})(\xi \otimes a) \\
&= (\mathrm{id}_{\mathcal{H}} \otimes \mathrm{id}_{C(\mathbb{G}_1)} \otimes \tilde{U}_0)(U \otimes \mathrm{id}_{L^2(\mathcal{B})} \otimes \mathrm{id}_{C(\mathbb{G}_2)})(\xi_{13} \otimes a) \\
&= (U \otimes \mathrm{id}_{L^2(\mathcal{B})} \otimes \mathrm{id}_{C(\mathbb{G}_2)})\big((\mathrm{id}_{\mathcal{H}} \otimes \tilde{U}_0)(\xi \otimes a)\big)_{134}
\end{aligned}$$

proving that indeed $\mathrm{id}_{\mathcal{H}} \otimes \tilde{U}_0$ restricts to a unitary element $\tilde{U} \in \mathcal{M}(\mathcal{K}(\mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})) \otimes C(\mathbb{G}_2))$.

Then it suffices to prove that $\tilde{U}$ commutes with the Dirac operator of the deformed spectral triple and that the action of $\mathbb{G}_2$ on $\tilde{\mathcal{A}}$ is algebraic (i.e. it is a Hopf algebraic coaction). First we check that $\tilde{U}_\phi$ leaves the domain of $\tilde{D}$ invariant for ϕ an arbitrary state on $C(\mathbb{G}_2)$. As $\tilde{U}$ is the restriction of $\mathrm{id}_{\mathcal{H}} \otimes \tilde{U}_0$ and β_1 and β_2 commute, $\tilde{U}$ restricts to a subrepresentation of $C(\mathbb{G}_2)$ on $V_\lambda \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})$. Hence, $\tilde{U}$ commutes with the spectral projections P'_λ of $\tilde{D}$. Then, as $\tilde{U}$ is unitary, it follows directly that

$$\begin{aligned}\sum_{\lambda\in\sigma(D)} |\lambda|^2 \|P'_\lambda \tilde{U}_\phi(\xi)\|^2 &= \sum_{\lambda\in\sigma(D)} |\lambda|^2 \|\tilde{U}_\phi P'_\lambda(\xi)\|^2 \\ &\leq \sum_{\lambda\in\sigma(D)} |\lambda|^2 \|P'_\lambda(\xi)\|^2 \\ &= \|\tilde{D}(\xi)\|^2\end{aligned}$$

where $P'_\lambda = P_\lambda \otimes \mathrm{id}$ are the spectral projections on the eigenspaces $V_\lambda \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})$. This proves that $\tilde{U}_\phi$ leaves the domain of $\tilde{D}$ invariant. As $\tilde{U}$ commutes with the spectral projections of $\tilde{D}$ it is then trivial that $\tilde{U}$ and $\tilde{D}$ themselves commute. Finally from theorem 2.6.10, we know that, given the coaction

$$\alpha_1 = \mathrm{ad}_U : \mathcal{A} \to \mathcal{A} \odot \mathcal{O}(\mathbb{G}_1) : a \to U(a \otimes \mathrm{id}_{\mathcal{A}})U^*,$$

there is a coaction $\alpha_2 : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}} \odot \mathcal{O}(\mathbb{G}_2) : z \to (\mathrm{id}_{\mathcal{A}} \otimes \beta_2)(z)$. Using (3.3.2), $\alpha_2 = \mathrm{id}_{\mathcal{A}} \otimes \mathrm{ad}_{\tilde{U}_0}$ and looking at elements of $\mathcal{A}$ as operators on $\mathcal{H}$, we have $\alpha_2 = \mathrm{ad}_{\tilde{U}}$ which is by construction a Hopf algebraic coaction of $\mathcal{O}(\mathbb{G}_2)$. □

Putting everything together, we have proven the main theorem of this chapter:

Theorem 3.3.8. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a compact spectral triple and let $\mathbb{G}_1 = (C(\mathbb{G}_1), \Delta_1)$ be a compact quantum group acting algebraically and by orientation-preserving isometries on $(\mathcal{A}, \mathcal{H}, D)$ with a unitary representation U. Moreover let ψ be a unitary fiber functor on $\mathbb{G}_1$.*

Then there exist a spectral triple $(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$, a compact quantum group $\mathbb{G}_2 = (C(\mathbb{G}_2), \Delta_2)$ monoidally equivalent with $\mathbb{G}_1$ and a unitary representation $\tilde{U}$ of $\mathbb{G}_2$ on $\tilde{\mathcal{H}}$ such that the monoidal equivalence is associated to ψ and $\mathbb{G}_2$ acts algebraically and by orientation-preserving isometries on the new spectral triple with $\tilde{U}$.

Denoting by $\mathcal{B}$ the ($\mathbb{G}_1$-$\mathbb{G}_2$)-bi-Galois object, one has

$$\tilde{\mathcal{A}} = \mathcal{A} \underset{\mathcal{O}(\mathbb{G}_1)}{\boxdot} \mathcal{B}, \qquad \tilde{\mathcal{H}} = \mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}), \qquad \tilde{D} = (D \otimes \mathrm{id}_{L^2(\mathcal{B})})_{|\tilde{\mathcal{H}}}. \tag{3.3.3}$$

In what follows, we will call this deformation procedure 'monoidal deformation'.

Finally, to be complete, we prove that our procedure is in fact a deformation, i.e. it is possible to get back to the original spectral triple via the inverse monoidal equivalence.

Theorem 3.3.9. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple, $\mathbb{G}_1$ a compact quantum group acting algebraically and by orientation-preserving isometries on $(\mathcal{A}, \mathcal{H}, D)$. Let ψ be a unitary fiber functor on $\mathbb{G}_1$ inducing a monoidal equivalence $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$ with bi-Galois object $\mathcal{B}$. Denote by $\varphi^{-1} : \mathbb{G}_2 \to \mathbb{G}_1$ the inverse monoidal equivalence with bi-Galois object $\tilde{\mathcal{B}}$. Then*

$$(\mathcal{A}, \mathcal{H}, D) \cong \left(\mathcal{A} \underset{\mathcal{O}(\mathbb{G}_1)}{\boxdot} \mathcal{B} \underset{\mathcal{O}(\mathbb{G}_2)}{\boxdot} \tilde{\mathcal{B}}, \mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}) \underset{C(\mathbb{G}_2)}{\boxtimes} L^2(\tilde{\mathcal{B}}), \tilde{\tilde{D}}\right)$$

where $\tilde{\tilde{D}} = (D \otimes \mathrm{id}_{L^2(\mathcal{B})} \otimes \mathrm{id}_{L^2(\tilde{\mathcal{B}})})|_{\mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}) \underset{C(\mathbb{G}_2)}{\boxtimes} L^2(\tilde{\mathcal{B}})}$. Moreover, the actions are isomorphic.

Proof. We recall first the notations of proposition 2.6.11. Let $\tilde{\mathcal{B}}$ be generated by the matrix coefficients of unitaries $Z^y \in B(\mathcal{H}_{\varphi^{-1}(y)}, \mathcal{H}_y) \odot \tilde{\mathcal{B}}$, $y \in \mathrm{Irred}(\mathbb{G}_2)$ and we call the coactions $\delta_1 : \tilde{B} \to \tilde{\mathcal{B}} \odot \mathcal{O}(\mathbb{G}_1)$ and $\delta_2 : \tilde{\mathcal{B}} \to \mathcal{O}(\mathbb{G}_2) \odot \tilde{\mathcal{B}}$ such that

$$(\mathrm{id} \otimes \delta_1) Z^y = Z^y_{12} U^{\varphi^{-1}(y)}_{13} \qquad \text{and} \qquad (\mathrm{id} \otimes \delta_2) Z^y = U^y_{12} Z^y_{13}.$$

Moreover we have a *-isomorphism

$$\pi : \mathcal{O}(\mathbb{G}_1) \to \mathcal{B} \underset{\mathcal{O}(\mathbb{G}_2)}{\boxdot} \tilde{\mathcal{B}} \qquad \text{determined by} \qquad (\mathrm{id} \otimes \pi)(U^x) = X^x_{12} Z^{\varphi(x)}_{13}.$$

Now $\alpha_U : \mathcal{A} \to \mathcal{A} \underset{C(\mathbb{G}_1)}{\boxdot} C(\mathbb{G}_1)$ is a *-isomorphism as well (with inverse $(\mathrm{id}_{\mathcal{A}} \otimes \varepsilon)$) and hence, one obtains the *-isomorphisms:

$$\lambda : \mathcal{A} \overset{\alpha_U}{\to} \mathcal{A} \underset{\mathcal{O}(\mathbb{G}_1)}{\boxdot} \mathcal{O}(\mathbb{G}_1) \overset{\mathrm{id} \otimes \pi}{\to} \mathcal{A} \underset{\mathcal{O}(\mathbb{G}_1)}{\boxdot} \mathcal{B} \underset{\mathcal{O}(\mathbb{G}_2)}{\boxdot} \tilde{\mathcal{B}}$$

such that

$$\begin{aligned}
(\lambda \otimes \mathrm{id}_{C(\mathbb{G}_1)}) \alpha_U &= ((\mathrm{id}_{\mathcal{A}} \otimes \pi) \alpha_U \otimes \mathrm{id}_{C(\mathbb{G}_1)}) \alpha_U \\
&= (\mathrm{id}_{\mathcal{A}} \otimes (\pi \otimes \mathrm{id}_{C(\mathbb{G}_1)}) \Delta_1) \alpha_U \\
&= (\mathrm{id}_{\mathcal{A}} \otimes (\mathrm{id}_{\mathcal{B}} \otimes \delta_1) \pi) \alpha_U \\
&= (\mathrm{id} \otimes \mathrm{id} \otimes \delta_1) \lambda.
\end{aligned}$$

Furthermore, recall the unitaries

$$f_x^{\varphi} : \mathcal{H}_{\varphi(x)} \to \mathcal{H}_x \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}) : \xi^{\varphi(x)} \mapsto X^x(\xi^{\varphi(x)} \otimes \Lambda(1_{\mathcal{B}}))$$

for $x \in \mathrm{Irred}(\mathbb{G}_1)$ of proposition 3.3.2. Note that these unitaries intertwine the representations of $\mathbb{G}_2$ on the two Hilbert spaces. We then also have

$$f_{\varphi(x)}^{\varphi^{-1}} : \mathcal{H}_x \to \mathcal{H}_{\varphi(x)} \underset{C(\mathbb{G}_2)}{\boxtimes} L^2(\tilde{\mathcal{B}}) : \eta^x \mapsto Z^{\varphi(x)}(\eta^x \otimes \tilde{\Lambda}(1_{\tilde{\mathcal{B}}}))$$

and combining them, we have a unitary:

$$\theta_x : \mathcal{H}_x \to \mathcal{H}_x \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}) \underset{C(\mathbb{G}_2)}{\boxtimes} L^2(\tilde{\mathcal{B}}) : \eta^x \mapsto X_{12}^x Z_{13}^{\varphi(x)}(\eta^x \otimes \Lambda(1_{\mathcal{B}}) \otimes \tilde{\Lambda}(1_{\tilde{\mathcal{B}}})).$$

Denoting by X and Z resp. $\oplus_{x \in \mathrm{Irred}(\mathbb{G}_1)} X^x$ and $\oplus_{x \in \mathrm{Irred}(\mathbb{G}_1)} Z^{\varphi(x)}$ (where we take the direct sum over the decomposition $\mathcal{H} = \oplus_{x \in \mathrm{Irred}(\mathbb{G}_1)} \mathcal{H}_x$), we then have a unitary

$$\theta = \oplus_{x \in \mathrm{Irred}(\mathbb{G}_1)} \theta_x : \mathcal{H} \to \mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}) \underset{C(\mathbb{G}_2)}{\boxtimes} L^2(\tilde{\mathcal{B}}) :$$

$$\xi \mapsto X_{12} Z_{13}(\xi \otimes \Lambda(1_{\mathcal{B}}) \otimes \tilde{\Lambda}(1_{\tilde{\mathcal{B}}}))$$

and hence

$$\begin{aligned} \lambda(a)\theta(\xi) &= (\mathrm{id} \otimes \pi)(\alpha_U(a))\theta(\xi) && (3.3.4) \\ &= (\mathrm{id} \otimes \pi)(U(a \otimes 1_{C(\mathbb{G}_1)})U^*) X_{12} Z_{13}(\xi \otimes \Lambda(1_{\mathcal{B}}) \otimes \tilde{\Lambda}(1_{\tilde{\mathcal{B}}})) \\ &= X_{12} Z_{13}(a \otimes 1_{\mathcal{B}} \otimes 1_{\tilde{\mathcal{B}}}) Z_{13}^* X_{12}^* X_{12} Z_{13}(\xi \otimes \Lambda(1_{\mathcal{B}}) \otimes \tilde{\Lambda}(1_{\tilde{\mathcal{B}}})) \\ &= X_{12} Z_{13}(a\xi \otimes \Lambda(1_{\mathcal{B}}) \otimes \tilde{\Lambda}(1_{\tilde{\mathcal{B}}})) \\ &= \theta(a\xi) && (3.3.5) \end{aligned}$$

where we used that $(\mathrm{id} \otimes \pi)U = X_{12} Z_{13}$ as in proposition 2.6.11. Furthermore, as $U(D \otimes \mathrm{id}_{C(\mathbb{G})}) = (D \otimes \mathrm{id}_{C(\mathbb{G})})U$ also $(\mathrm{id}_{\mathcal{H}} \otimes \pi)U(D \otimes \mathrm{id}_{C(\mathbb{G})}) = (D \otimes \mathrm{id}_{C(\mathbb{G})})(\mathrm{id}_{\mathcal{H}} \otimes \pi)U$ and hence we have $\theta(V_\lambda) \subset V_\lambda \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}) \underset{C(\mathbb{G}_2)}{\boxtimes} L^2(\tilde{\mathcal{B}})$ and hence $\theta \circ D = \tilde{\tilde{D}} \circ \theta$. Finally, we have to check that the representations of $\mathbb{G}_1$ are intertwined by θ. We have

$$\begin{aligned}
&(\theta\otimes \mathrm{id}_{C(\mathbb{G}_1)})U(\xi\otimes 1_{C(\mathbb{G}_1)})\\
&= X_{12}Z_{13}U_{14}(\xi\otimes\Lambda(1_{\mathcal{B}})\otimes\tilde{\Lambda}(1_{\tilde{\mathcal{B}}})\otimes 1_{C(\mathbb{G}_1)})\\
&= X_{12}(\mathrm{id}_{\mathcal{H}}\otimes 1_{\mathcal{B}}\otimes\delta_2)(Z_{13})(\xi\otimes\Lambda(1_{\mathcal{B}})\otimes\tilde{\Lambda}(1_{\tilde{\mathcal{B}}})\otimes 1_{C(\mathbb{G}_1)})\\
&= (\mathrm{id}_{\mathcal{H}}\otimes\mathrm{id}_{L^2(\mathcal{B})}\otimes\delta_2)(X_{12}Z_{13})(\xi\otimes\Lambda(1_{\mathcal{B}})\otimes\tilde{\Lambda}(1_{\tilde{\mathcal{B}}})\otimes 1_{C(\mathbb{G}_1)})\\
&= \widetilde{(\tilde{U})}(\theta(\xi)\otimes 1_{C(\mathbb{G}_1)})
\end{aligned}$$

concluding the proof. □

3.4 Conclusion

In this third chapter we proved the main result of this thesis. Starting form a spectral triple of compact type with a compact quantum group acting algebraically and by orientation preserving isometries on it and a unitary fiber functor on the quantum group, we can construct a new spectral triple on which the new compact quantum group induced by the unitary fiber functor acts also algebraically and by orientation preserving isometries.

Appendix on unbounded operators

In this appendix, we give some theory about unbounded operators, with a focus on the tensor product of an unbounded operator with the identity operator on a Hilbert space. In this chapter, we use the material of Schmüdgen's 'Unbounded Self-adjoint Operators on Hilbert Space' [89].

Remark 3.4.1. *Let T be an unbounded operator. We will denote by $\mathcal{D}(T)$ the domain of T, by $\sigma(T)$ the spectrum of T and by $\rho(T)=\mathbb{C}\setminus\sigma(T)$ the resolvent.*

Definition 3.4.2. *Let T be an unbounded operator.*

- *For an element $\lambda\in\rho(T)$ we define*

$$R_\lambda(T)=(T-\lambda I)^{-1}.$$

- *T is said to have compact resolvent if $R_\lambda(T)$ is compact for all $\lambda \in \rho(T)$.*
- *T is said to have purely point spectrum if every element of the spectrum is an eigenvalue of finite multiplicity which has no finite accumulation point.*

Proposition 3.4.3 (Proposition 5.12 in [89]). *Let T be a self adjoint unbounded operator on an infinite dimensional Hilbert space $\mathcal{H}$. Then the following are equivalent*

- *T has purely point spectrum,*
- *$R_\lambda(T)$ is a compact operator for one and hence for all $\lambda \in \rho(T)$,*
- *There exist a real sequence $(\lambda_n)_{n\in\mathbb{N}}$ and an orthonormal basis $\{e_n | n \in \mathbb{N}\}$ of $\mathcal{H}$ such that $\lim_{n\to\infty} |\lambda_n| = \infty$ and $Te_n = \lambda_n e_n$.*

In this section, T_1 and T_2 denote unbounded operators on Hilbert spaces $\mathcal{H}_1$ resp. $\mathcal{H}_2$. Define

$$\mathcal{D}(T_1 \odot T_2) := \langle x \otimes y | x \in \mathcal{D}(T_1), y \in \mathcal{D}(T_2)\rangle$$

and

$$(T_1 \odot T_2)(\sum_{k=1}^{n} x_k \otimes y_k) = \sum_{k=1}^{n} T_1(x_k) \odot T_2(y_k)$$

for $x_k \in \mathcal{D}(T_1), y_k \in \mathcal{D}(T_2), n \in \mathbb{N}$.

Proposition 3.4.4 (Proposition 7.20 in [89]).

1. *$T_1 \odot T_2$ is a well-defined operator on the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ with domain $\mathcal{D}(T_1 \odot T_2)$.*
2. *If T_1 and T_2 are bounded, the, so is $T_1 \odot T_2$ and $\|T_1 \odot T_2\| = \|T_1\|\|T_2\|$.*

Proposition 3.4.5 (Lemma 7.21 and Definition 7.3 in [89]). *Let T_1, T_2 be two densely defined and closable operators. Then $T_1 \odot T_2$ is also densely defined and closable. The closure of the closable operator $T_1 \odot T_2$ is denoted by $T_1 \otimes T_2$ and called the tensor product of T_1 and T_2.*

Proposition 3.4.6 (Theorem 7.23 and Corollary 7.25 in [89]). *Suppose T is selfadjoint, then $T \otimes 1$ is well defined and selfadjoint and $\sigma(T \otimes 1) = \sigma(T)$. If T is positive, so is $T \otimes 1$.*

Proposition 3.4.7 (Corollary 3.14 in [89]). *Let T be a closed symmetric linear operator, then T is selfadjoint if and only if $\sigma(T) \subset \mathbb{R}$.*

Chapter 4

2-Cocycle deformation of spectral triples

In this chapter we have a closer look at unitary fiber functors and monoidal equivalences with an extra property. We recalled in section 2.6 that monoidal equivalences preserve the quantum dimensions, i.e. for a monoidal equivalence $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$, $\dim_q(\varphi(x)) = \dim_q(x)$ for every irreducible representation x of $\mathbb{G}_1$. In this chapter we investigate unitary fiber functors ψ (or equivalently monoidal equivalences) with the extra property that $\dim(\psi(x)) = \dim(x)$. Unitary fiber functors which satisfy this condition will be called dimension-preserving. A monoidal deformation arising from a dimension-preserving unitary fiber functor is called a dimension-preserving monoidal deformation. Bichon et al. proved in [27] that dimension-preserving unitary fiber functors are in one-to-one correspondence with 2-cocycles on the dual quantum group. Using this, we will prove that dimension-preserving monoidal deformation is equivalent to the cocycle deformation introduced in [53]. In this chapter we will frequently use slight adaptations of [27].

This chapter is structured as follows. In the first section, we recall the notion of a 2-cocycle on the dual $\hat{\mathbb{G}}$ of a compact quantum group $\mathbb{G}$ and state the result of [27] that 2-cocycles on $\hat{\mathbb{G}}$ are in one-to-one correspondence with dimension-preserving unitary fiber functors. In the second section, we remind what algebraic dual 2-cocycles are and state the deformation procedure of Goswami and Joardar [53]. In section 3, we make the link between algebraic and analytical 2-cocycles and in in the fourth and last section, we prove the equivalence of the deformation à

la Goswami-Joardar and the dimension-preserving monoidal deformation. Finally, in the appendix, we correct two errors. First, the Goswami-Joardar paper refers to work of Majid ([71]) about real 2-cocycles. However, [53] works with unitary 2-cocycles and the deformation of a Hopf *-algebra with a unitary 2-cocycle is not yet defined. We do that in the first part of the appendix.
In the second part, we focus on the work of Majid in [71], to which [53] refers. In fact, this theorem in [71] contains an error, which we correct there.

4.1 Cocycles on the dual of a compact quantum group

In this section we describe unitary cocycles on the dual of a compact quantum group and investigate the link with dimension-preserving unitary fiber functors.

Definition 4.1.1. [1] *Let $\mathbb{G}$ be a compact quantum group and $(c_0(\hat{\mathbb{G}}), \hat{\Delta})$ its dual. We say a unitary element $\Omega \in \mathcal{M}(c_0(\hat{\mathbb{G}}) \otimes c_0(\hat{\mathbb{G}}))$ is a 2-cocycle on $\hat{\mathbb{G}}$ if it satisfies*

$$(\Omega \otimes 1)(\hat{\Delta} \otimes \mathrm{id})(\Omega) = (1 \otimes \Omega)(\mathrm{id} \otimes \hat{\Delta})(\Omega). \tag{4.1.1}$$

Denoting by p_x the projection $c_0(\hat{\mathbb{G}}) \to B(\mathcal{H}_x)$ for $x \in \mathrm{Irred}(\mathbb{G})$ and by ε the class of the trivial representation, we will say a cocycle is normalized if $(p_\varepsilon \otimes \mathrm{id})\Omega = p_\varepsilon \otimes \mathrm{id}$ and $(\mathrm{id} \otimes p_\varepsilon)\Omega = \mathrm{id} \otimes p_\varepsilon$. From now on we will always assume 2-cocycles to be normalized.

Proposition 4.1.2 ([27]). *Let Ω be a normalized unitary 2-cocycle on $\hat{\mathbb{G}}$ and denote*

$$\Omega_{(2)} = (\Omega \otimes 1)(\hat{\Delta} \otimes \mathrm{id})(\Omega) = (1 \otimes \Omega)(\mathrm{id} \otimes \hat{\Delta})(\Omega).$$

Then there exists a unique unitary fiber functor ψ_Ω on $\mathbb{G}$ such that

$$\mathcal{H}_{\psi_\Omega(x)} = \mathcal{H}_x, \qquad \psi_\Omega(R) = R, \qquad \psi_\Omega(S) = \Omega S, \qquad \psi_\Omega(T) = \Omega_{(2)} T$$

for all $R \in \mathrm{Mor}(y, x)$, $S \in \mathrm{Mor}(y \otimes z, x)$ and $T \in \mathrm{Mor}(x \otimes y \otimes z, a)$ where $a, x, y, z \in \mathrm{Irred}(\mathbb{G})$. By construction it is dimension-preserving.

Proof. The proof follows directly as our ψ satisfies the conditions of remark 2.6.2. Indeed, $\psi_\Omega(1) = 1$, for $S \in \mathrm{Mor}(x \otimes y, a), T \in \mathrm{Mor}(x \otimes y, b)$,

$$\psi_\Omega(S)^* \psi_\Omega(T) = (S^* \Omega^*)(\Omega T) = S^* T = \psi_\Omega(S^* T),$$

[1] In [27], the authors use another convention for cocycle. In fact, if Ω is a cocycle in our sense, Ω^* is one in the sense of Bichon et al.

for $S \in \mathrm{Mor}(x \otimes y, a), T \in \mathrm{Mor}(a \otimes z, b)$,

$$(\psi_\Omega(S) \otimes \mathrm{id})\psi_\Omega(T) = (\Omega S \otimes \mathrm{id})(\Omega T) = (\Omega \otimes \mathrm{id})(\hat{\Delta} \otimes \mathrm{id})\Omega(S \otimes \mathrm{id})T$$
$$= \Omega_{(2)}(S \otimes \mathrm{id})T = \psi_\Omega\big((S \otimes \mathrm{id})T\big)$$

and for $S \in \mathrm{Mor}(y \otimes z, a), T \in \mathrm{Mor}(x \otimes a, b)$,

$$(\mathrm{id} \otimes \psi_\Omega(S))\psi_\Omega(T) = (\mathrm{id} \otimes \Omega S)\Omega T = (\mathrm{id} \otimes \Omega)(\mathrm{id} \otimes \hat{\Delta})\Omega(\mathrm{id} \otimes S)T$$
$$= \Omega_{(2)}(\mathrm{id} \otimes S)T = \psi_\Omega\big((\mathrm{id} \otimes S)T\big).$$

Moreover, as Ω is unitary and

$$[S\xi | x \in \mathrm{Irred}(\mathbb{G}), S \in \mathrm{Mor}(y \otimes z, x), \xi \in \mathcal{H}_x] = \mathcal{H}_y \otimes \mathcal{H}_z$$

for $y, z \in \mathrm{Irred}(\mathbb{G})$, also

$$[\psi_\Omega(S)\xi | x \in \mathrm{Irred}(\mathbb{G}), S \in \mathrm{Mor}(y \otimes z, x), \xi \in \mathcal{H}_{\psi(x)}] = \mathcal{H}_{\psi(y)} \otimes \mathcal{H}_{\psi(z)}$$

for $y, z \in \mathrm{Irred}(\mathbb{G})$ and hence ψ_Ω is a well defined unitary fiber functor. □

Using this unitary fiber functor, one can make a new compact quantum group $\mathbb{G}_\Omega = (C(\mathbb{G}_\Omega), \Delta_\Omega)$ [27] and a monoidal equivalence $\varphi : \mathbb{G} \to \mathbb{G}_\Omega$ along the lines of proposition 2.6.4. Note that the dual quantum group will be $(c_0(\hat{\mathbb{G}}_\Omega), \hat{\Delta}_\Omega)$ where

$$c_0(\hat{\mathbb{G}}_\Omega) = \bigoplus_{x \in \mathrm{Irred}(\mathbb{G})} B(\mathcal{H}_x) = c_0(\hat{\mathbb{G}})$$

and

$$\hat{\Delta}_\Omega(a)\psi_\Omega(S) = \psi_\Omega(S)a$$

where $\hat{\Delta}_\Omega(a) = \Omega\Delta(a)\Omega^*$ for $a \in B(\mathcal{H}_x), S \in \mathrm{Mor}(y \otimes z, x), x, y, z \in \mathrm{Irred}(\mathbb{G})$.

There is even more: every dimension-preserving unitary fiber functor is of this form.

Proposition 4.1.3 ([27]). *For every dimension-preserving unitary fiber functor ψ on a compact quantum group $\mathbb{G}$, there exists a normalized unitary 2-cocycle Ω on $\hat{\mathbb{G}}$ such that $\psi \cong \psi_\Omega$.*

Proof. Denoting $\varphi : \mathbb{G} \to \mathbb{G}_\Omega$ to be the monoidal equivalence associated to ψ, we can find unitaries $u_x = \mathcal{H}_x \to \mathcal{H}_{\varphi(x)}$, as $\dim(\varphi(x)) = \dim(x)$ for all $x \in \mathrm{Irred}(\mathbb{G})$. Fixing a $x \in \mathrm{Irred}(\mathbb{G})$, we can define $Y^x = X^x(u_x \otimes 1) \in B(\mathcal{H}_x) \odot \mathcal{B}$ and

$$Y' = \oplus_{x \in \mathrm{Irred}(\mathbb{G})} Y^x \in \mathcal{M}(c_0(\hat{\mathbb{G}}) \otimes B_r)$$

(where we take the direct sum over all classes, all of them with multiplicity one). Note that Y' is unitary by construction.

Now, we have that

$$(\mathrm{id} \otimes \beta_1) Y' = \bigoplus_{x \in \mathrm{Irred}(\mathbb{G})} (\mathrm{id} \otimes \beta_1) X^x (u_x \otimes \mathrm{id}) = \bigoplus_{x \in \mathrm{Irred}(\mathbb{G})} U^x_{12} X^x_{13} (u_x \otimes \mathrm{id}) = \mathbb{V}_{12} Y'_{13}$$

where $\mathbb{V} = \oplus_{x \in \mathrm{Irred}(\mathbb{G})} U^x$ and hence

$$\begin{aligned}
&(\mathrm{id} \otimes \mathrm{id} \otimes \beta_1)((Y'_{23})^* (Y'_{13})^* (\hat{\Delta} \otimes \mathrm{id})(Y')) \\
&= (Y'_{24})^* \mathbb{V}^*_{23} (Y'_{14})^* \mathbb{V}^*_{13} (\hat{\Delta} \otimes \mathrm{id} \otimes \mathrm{id})(\mathbb{V}_{12} Y'_{13}) \\
&= (Y'_{24})^* (Y'_{14})^* \mathbb{V}^*_{23} \mathbb{V}^*_{13} \mathbb{V}_{13} \mathbb{V}_{23} (\hat{\Delta} \otimes \mathrm{id} \otimes \mathrm{id})(Y'_{13}) \\
&= (Y'_{24})^* (Y'_{14})^* (\hat{\Delta} \otimes \mathrm{id} \otimes \mathrm{id})(Y'_{13})
\end{aligned}$$

which means that $(Y'_{23})^*(Y'_{13})^*(\hat{\Delta} \otimes \mathrm{id})(Y')$ is invariant under $(\mathrm{id} \otimes \mathrm{id} \otimes \beta_1)$ and, as β_1 is ergodic, there exists an element $\Omega \in \mathcal{M}(c_0(\hat{\mathbb{G}}) \otimes c_0(\hat{\mathbb{G}}))$ such that $(Y'_{23})^*(Y'_{13})^*(\hat{\Delta} \otimes \mathrm{id})(Y') = \Omega \otimes 1_{\mathcal{B}}$ and hence

$$(\hat{\Delta} \otimes \mathrm{id})(Y') = Y'_{13} Y'_{23} (\Omega \otimes 1_{\mathcal{B}}). \tag{4.1.2}$$

As Y' is unitary, so is Ω. Moreover, we have

$$\begin{aligned}
(\mathrm{id} \otimes \hat{\Delta} \otimes \mathrm{id})(\hat{\Delta} \otimes \mathrm{id})(Y') &= (\mathrm{id} \otimes \hat{\Delta} \otimes \mathrm{id})(Y'_{13} Y'_{23} (\Omega \otimes 1_{\mathcal{B}})) \\
&= Y'_{14} Y'_{24} Y'_{34} (\mathrm{id} \otimes \Omega \otimes 1_{\mathcal{B}})((\mathrm{id} \otimes \hat{\Delta})\Omega \otimes 1_{\mathcal{B}})
\end{aligned}$$

and

$$\begin{aligned}
(\hat{\Delta} \otimes \mathrm{id} \otimes \mathrm{id})(\hat{\Delta} \otimes \mathrm{id})(Y') &= (\hat{\Delta} \otimes \mathrm{id} \otimes \mathrm{id})(Y'_{13} Y'_{23} (\Omega \otimes 1_{\mathcal{B}})) \\
&= Y'_{14} Y'_{24} Y'_{34} (\Omega \otimes \mathrm{id} \otimes 1_{\mathcal{B}})((\hat{\Delta} \otimes \mathrm{id})\Omega \otimes 1_{\mathcal{B}})
\end{aligned}$$

and using coassociativity, we see that Ω is a unitary 2-cocycle. To end this proof, we have to show that φ is isomorphic to φ_Ω. Therefore, note that for $S \in \mathrm{Mor}(y \otimes z, x)$, $(S \otimes \mathrm{id})(Y^x)^* = (\hat{\Delta} \otimes \mathrm{id})(Y')^*(S \otimes \mathrm{id})$ and hence

$$
\begin{aligned}
(\varphi_\Omega(S) \otimes \mathrm{id}) = (\Omega S \otimes \mathrm{id}) &= (\Omega \otimes \mathrm{id})(\hat{\Delta} \otimes \mathrm{id})(Y')^*(S \otimes \mathrm{id})Y^x \\
&= (Y'_{23})^*(Y'_{13})^*(S \otimes \mathrm{id})X^x(u_x \otimes 1) \\
&= (Y'_{23})^*(Y'_{13})^* X^y_{13} X^z_{23}(\varphi(S) \otimes \mathrm{id})(u_x \otimes 1) \\
&= (u_y^* \otimes u_z^* \otimes \mathrm{id})(\varphi(S) \otimes \mathrm{id})(u_x \otimes \mathrm{id})
\end{aligned}
$$

which confirms that indeed φ is isomorphic with φ_Ω.

□

This theorems tells us that every dimension-preserving monoidal equivalence comes from a unitary cocycle on the dual quantum group.

4.2 Algebraic 2-cocycle deformation of a spectral triple

In this section we recall the notion of algebraic dual 2-cocycles and remind the deformation procedure introduced by Goswami and Joardar in [53].

4.2.1 Algebraic 2-cocycles

We will start with defining the algebraic counterpart of a 2-cocycle on the dual of a compact quantum group. In algebraic literature (for example [88]), the definition and theorems are stated for Hopf algebras. We make slight adaptations to Hopf *-algebras.

Definition 4.2.1. *Let H be a Hopf-algebra.*

1. *An (algebraic) dual 2-cocycle on H is a linear map $\sigma : H \odot H \to \mathbb{C}$ such that*

$$\sigma(a_{(1)}, b_{(1)})\sigma(a_{(2)}b_{(2)}, c) = \sigma(b_{(1)}, c_{(1)})\sigma(a, b_{(2)}c_{(2)})$$

for all $a, b, c \in H$. It is called normalized if $\sigma(1, h) = \sigma(h, 1) = \varepsilon(h)$ for all $h \in H$.

2. *A dual 2-cocycle is called invertible if there exists a linear map $\sigma' : H \odot H \to \mathbb{C}$ such that*

$$\sigma(a_{(1)}, b_{(1)})\sigma'(a_{(2)}, b_{(2)}) = \varepsilon(a)\varepsilon(b) = \sigma'(a_{(1)}, b_{(1)})\sigma(a_{(2)}, b_{(2)}).$$

In this case, σ' is unique and it is called the inverse dual cocycle and written σ^{-1}. Moreover σ^{-1} satisfies

$$\sigma^{-1}(a_{(1)}b_{(1)}, c)\sigma^{-1}(a_{(2)}, b_{(2)}) = \sigma^{-1}(a, b_{(1)}c_{(1)})\sigma^{-1}(b_{(2)}, c_{(2)}).$$

3. *If H is a Hopf *-algebra, a dual 2-cocycle σ is called unitary if it satisfies*

$$\overline{\sigma(a, b)} = \sigma^{-1}(S(a)^*, S(b)^*).$$

In that case, we also have

$$\overline{\sigma^{-1}(a, b)} = \sigma(S(a)^*, S(b)^*).$$

In the rest of the chapter, when we use dual 2-cocycles on Hopf *-algebras, we will always assume them to be unitary.

Using such a dual 2-cocycle, we can make a new *-algebra and several new H-comodule-*-algebras. We will use the following linear maps:

- $U : H \to \mathbb{C} : h \mapsto \sigma(h_{(1)}, S(h_{(2)}))$,
- $V : H \to \mathbb{C} : h \mapsto U(S^{-1}(h))$.

One can prove that using the notations

$$U^{-1}(h) = \sigma^{-1}(S(h_{(1)}), h_{(2)}) \quad \text{and} \quad V^{-1}(h) = U^{-1}(S^{-1}(h))$$

one has

$$U(h_{(1)})U^{-1}(h_{(2)}) = \varepsilon(h) = U^{-1}(h_{(1)})U(h_{(2)})$$

and

$$V(h_{(1)})V^{-1}(h_{(2)}) = \varepsilon(h) = V^{-1}(h_{(1)})V(h_{(2)}).$$

Note that this inverse notation should not be confused with the inverses of the maps U or V, but is meant to be the convolution product inverse.

With a dual 2-cocycle on a Hopf algebra, one can make a new Hopf algebra, and two bi-comodule-algebras.

Definition 4.2.2 ([71]). *Given an invertible dual 2-cocycle σ on a Hopf algebra $(H, \Delta, \varepsilon, S)$, we define $(H^\sigma, \Delta_\sigma, \varepsilon_\sigma, S_\sigma)$ to be the Hopf algebra which*

- *is isomorphic to H as a coalgebra,*
- *has multiplication defined by $g \cdot_\sigma h = \sigma(g_{(1)}, h_{(1)}) g_{(2)} h_{(2)} \sigma^{-1}(g_{(3)}, h_{(3)})$,*
- *has antipode $S_\sigma(h) = U(h_{(1)}) S(h_{(2)}) U^{-1}(h_{(3)})$,*
- *has counit $\varepsilon_\sigma = \varepsilon$.*

It is called the twisted Hopf algebra.

This definition was already given by Majid [71]. If H is a Hopf *-algebra and σ a unitary cocycle, we can state a new result:

Proposition 4.2.3. *Let H be a Hopf *-algebra and σ an invertible unitary dual 2-cocycle σ on H. Then the involution $h^{*_\sigma} = V^{-1}(h^*_{(1)}) h^*_{(2)} V(h^*_{(3)})$ makes the twisted Hopf algebra $(H^\sigma, \Delta_\sigma, \varepsilon_\sigma, S_\sigma)$ a Hopf *-algebra.*

Proof. We prove this proposition in the appendix of this chapter. □

Also the existence of a $(H^\sigma$-$H)$- and a $(H$-$H^\sigma)$-bi-comodule-algebra was already proven in [71].

Definition 4.2.4. *Let H be a Hopf algebra and σ an invertible dual 2-cocycle. We define*

1. *$\mathbb{C}\#_\sigma H$ to be a (H^σ-H)-bi-comodule-algebra which*
   - *is isomorphic to H as right H-comodule,*
   - *has twisted multiplication $(1\#g)(1\#h) = \sigma(g_{(1)}, h_{(1)})\#g_{(2)}h_{(2)}$,*
   - *and has a coaction $\beta_1 : \mathbb{C}\#_\sigma H \to H^\sigma \odot (\mathbb{C}\#_\sigma H) : (1\#h) \mapsto h_{(1)} \otimes (1\#h_{(2)})$*

   and
2. *$H_{\sigma^{-1}}\#\mathbb{C}$ to be a (H-H^σ)-bi-comodule-algebra which*
   - *is isomorphic to H as left H-comodule,*
   - *has twisted multiplication $(g\#1)(h\#1) = g_{(1)}h_{(1)}\#\sigma^{-1}(g_{(2)}, h_{(2)})$,*

- *and has a coaction* $\beta_2 : H_{\sigma^{-1}}\#\mathbb{C} \to (H_{\sigma^{-1}}\#\mathbb{C}) \odot H^\sigma : (h\#1) \mapsto (h_{(1)}\#1) \otimes h_{(2)}$.

The extension to Hopf *-algebras, unitary dual 2-cocycles and bi-comodule *-algebras however is a new result.

Proposition 4.2.5. *Let H be a Hopf *-algebra and σ an invertible unitary dual 2-cocycle on H. Then the involution $(1\#h)^{*_{\mathbb{C}\#_\sigma H}} = 1\#V^{-1}(h^*_{(1)})h^*_{(2)}$ makes $\mathbb{C}\#_\sigma H$ a (H^σ-H)-bi-comodule *-algebra and the involution $(h\#1)^{*_{H_{\sigma^{-1}}\#\mathbb{C}}} = h^*_{(1)}V(h^*_{(2)})\#1$ makes $H_{\sigma^{-1}}\#\mathbb{C}$ a (H-H^σ)-bi-comodule *-algebra.*

Proof. We give the proof in the appendix. □

Proposition 4.2.6. *Let H be a Hopf *-algebra and σ an invertible unitary dual 2-cocycle on H. Then $\mathbb{C}\#_\sigma H$ is a (H^σ-H)-bi-Galois-object with inverse $H_{\sigma^{-1}}\#\mathbb{C}$ in the groupoid of bi-Galois objects.*

Proof. One can check that the map

$$T_{\beta_1} : \mathbb{C}\#_\sigma H \odot \mathbb{C}\#_\sigma H \to H^\sigma \odot \mathbb{C}\#_\sigma H : (1\#h)\otimes(1\#g) \mapsto h_{(1)} \otimes (1\#h_{(2)})(1\#g)$$

is a bijection with inverse

$$P_{\beta_1} : H^\sigma \odot \mathbb{C}\#_\sigma H \to \mathbb{C}\#_\sigma H \odot \mathbb{C}\#_\sigma H : h \otimes (1\#g) \mapsto \gamma_1(h)\big(1 \otimes (1\#g)\big)$$

where

$$\gamma_1 : H^\sigma \to \mathbb{C}\#_\sigma H \odot \mathbb{C}\#_\sigma H : h \mapsto (1\#h_{(1)}) \otimes \big(\sigma^{-1}(S(h_{(3)}), h_{(4)})\#S(h_{(2)})\big).$$

Indeed

$$\begin{aligned}
T_{\beta_1}(\gamma_1(h)) &= h_{(1)} \otimes (1\#h_{(2)})\big(\sigma^{-1}(S(h_{(4)}), h_{(5)})\#S(h_{(3)})\big) \\
&= h_{(1)} \otimes \big(\sigma^{-1}(S(h_{(6)}), h_{(7)})\sigma(h_{(2)}, S(h_{(5)}))\#h_{(3)}S(h_{(4)})\big) \\
&= h_{(1)} \otimes \big(\sigma^{-1}(S(h_{(4)}), h_{(5)})\sigma(h_{(2)}, S(h_{(3)}))\#1\big) \\
&= h \otimes (1\#1)
\end{aligned}$$

and hence

$$T_{\beta_1}(P_{\beta_1}(h \otimes (1\#g))) = T_{\beta_1}(\gamma_1(h))(1 \otimes (1\#g)) = h \otimes (1\#g).$$

Also

$$
\begin{aligned}
& P_{\beta_1}\Big(T_{\beta_1}\big((1\#h)\otimes(1\#g)\big)\Big) \\
= \quad & P_{\beta_1}\Big(h_{(1)}\otimes(1\#h_{(2)})(1\#g)\Big) \\
= \quad & \gamma_1(h_{(1)})\big(1\otimes(1\#h_{(2)})(1\#g)\big) \\
= \quad & (1\#h_{(1)})\otimes\big(\sigma^{-1}(S(h_{(3)}),h_{(4)})\#S(h_{(2)})\big)(1\#h_{(5)})(1\#g) \\
= \quad & (1\#h_{(1)})\otimes\big(\sigma^{-1}(S(h_{(4)}),h_{(5)})\sigma(S(h_{(3)}),h_{(6)})\#S(h_{(2)})h_{(7)}\big)(1\#g) \\
= \quad & (1\#h_{(1)})\otimes\big(1\#S(h_{(2)})h_{(3)}\big)(1\#g) \\
= \quad & (1\#h)\otimes(1\#g).
\end{aligned}
$$

Analogously one can prove that $R_{\beta_2}:\mathbb{C}\#_\sigma H\odot\mathbb{C}\#_\sigma H\to\mathbb{C}\#_\sigma H\odot H$ is bijective. Note moreover that the H- and H^σ-coactions commute and hence $\mathbb{C}\#_\sigma H$ is a $(H^\sigma$-$H)$-bi-Galois-object. To prove that $H_{\sigma^{-1}}\#\mathbb{C}$ is the inverse of $\mathbb{C}\#_\sigma H$ in the groupoid of bi-Galois objects, we define the following map:

$$G_\sigma:\mathbb{C}\#_\sigma H\to H_{\sigma^{-1}}\#\mathbb{C}:1\#h\mapsto V(h_{(1)})S^{-1}(h_{(2)})\#1.$$

It is easy to see that it is a linear map with inverse

$$H_{\sigma^{-1}}\#\mathbb{C}\to\mathbb{C}\#_\sigma H:g\#1\mapsto S(g_{(1)})V^{-1}(S(g_{(2)}))\#1.$$

One can check moreover that G_σ is an isomorphism from $\big(\mathbb{C}\#_\sigma H\big)^{op}$ to $H_{\sigma^{-1}}\#\mathbb{C}$ which is compatible with the respective coactions. Combining this with proposition 1.4.5, this proves that $H_{\sigma^{-1}}\#\mathbb{C}$ is the inverse of $\mathbb{C}\#_\sigma H$ in the groupoid of bi-Galois objects. □

The following definition is well known.

Definition 4.2.7. *Let H be a Hopf algebra and σ an invertible dual 2-cocycle on H. Let A be a right H-comodule-algebra with coaction $\alpha:A\to A\odot H$. We define $A_{\sigma^{-1}}\#\mathbb{C}$ to be a right H^σ-comodule *-algebra which*

- *is isomorphic to A as vector space,*
- *has multiplication $(a\#1)(a'\#1)=a_{(0)}a'_{(0)}\#\sigma^{-1}(a_{(1)},a'_{(1)})$,*

- *and has a coaction* $\tilde{\alpha} : A_{\sigma^{-1}}\#\mathbb{C} \to (A_{\sigma^{-1}}\#\mathbb{C}) \odot H^\sigma : (a\#1) \mapsto (a_{(0)}\#1) \otimes a_{(1)}$.

And again the extension to *-algebras is new:

Proposition 4.2.8. *Let H be a Hopf *-algebra and σ an invertible unitary dual 2-cocycle on H. Let A be a right H-comodule *-algebra with coaction $\alpha : A \to A \odot H$. Then $A_{\sigma^{-1}}\#\mathbb{C}$ is a a right H^σ-comodule *-algebra with involution $(a\#1)^{*_{A_{\sigma^{-1}}\#\mathbb{C}}} = a^*_{(0)}V(a^*_{(1)})\#1$.*

Proof. We give the proof in the appendix. □

Theorem 4.2.9. *Let H be a Hopf *-algebra and A a right H-comodule *-algebra with coaction $\alpha : A \to A \odot H$. Denote $B = H_{\sigma^{-1}}\#\mathbb{C}$. Then*

$$A \underset{H}{\boxdot} B \cong A_{\sigma^{-1}}\#\mathbb{C}$$

as right H^σ-comodule-*-algebras.

Proof. We have the natural *-algebraic isomorphisms

$$A \stackrel{\alpha}{\to} A \underset{H}{\boxdot} H \stackrel{\mathrm{id}\otimes\varepsilon}{\to} A.$$

Using it as vector space isomorphisms, deforming the multiplications and using that B and H are isomorphic as left H-comodules, it is easy to check that we have a well defined *-algebra isomorphism

$$\lambda : A_{\sigma^{-1}}\#\mathbb{C} \to A \underset{H}{\boxdot} B : (a\#1) \to a_{(0)} \otimes (a_{(1)}\#1).$$

Indeed, we have

$$\begin{aligned}
\lambda\big((a\#1)(a'\#1)\big) &= \lambda\big((a_{(0)}a'_{(0)}\#1)\big)\sigma^{-1}(a_{(1)}, a'_{(1)})) \\
&= a_{(0)}a'_{(0)} \odot (a_{(1)}a'_{(1)}\#1)\sigma^{-1}(a_{(2)}, a'_{(2)})) \\
&= a_{(0)}a'_{(0)} \odot (a_{(1)}\#1)(a'_{(1)}\#1) \\
&= \lambda(a\#1)\lambda(a'\#1)
\end{aligned}$$

and

$$\begin{aligned}
\lambda((a\#1)^{*_{A_{\sigma^{-1}}\#\mathbb{C}}}) &= \lambda(a^*_{(0)}V(a^*_{(1)})\#1) \\
&= a^*_{(0)} \otimes (a^*_{(1)}V(a^*_{(2)})\#1) \\
&= a^*_{(0)} \otimes (a_{(1)}\#1)^{*_B} \\
&= (a_{(0)} \otimes (a_{(1)}\#1))^{*_{A\odot B}} \\
&= \lambda(a)^{*_{A\odot B}}.
\end{aligned}$$

Moreover, denoting the coactions by

$$\beta_2 : H_{\sigma^{-1}}\#\mathbb{C} \to (H_{\sigma^{-1}}\#\mathbb{C}) \odot H^\sigma : (h\#1) \mapsto (h_{(1)}\#1) \otimes h_{(2)}$$

and

$$\tilde{\alpha} : A_{\sigma^{-1}}\#\mathbb{C} \to (A_{\sigma^{-1}}\#\mathbb{C}) \odot H^\sigma : (a\#1) \mapsto (a_{(0)}\#1) \otimes a_{(1)},$$

we have

$$\begin{aligned}
(\lambda \otimes \mathrm{id}_{H^\sigma})\tilde{\alpha}(a\#1) &= \lambda(a_{(0)}\#1) \otimes a_{(1)} \\
&= a_{(0)} \otimes (a_{(1)}\#1) \otimes a_{(2)} \\
&= a_{(0)} \otimes \beta_2(a_{(1)}\#1) \\
&= (\mathrm{id}_A \otimes \beta_2)\lambda(a\#1).
\end{aligned}$$

We can conclude that λ is an isomorphism of right H^σ-comodule-*-algebras. □

4.2.2 Algebraic 2-cocycle deformation as defined by Goswami - Joardar

In this subsection we give a slightly adapted version of the main result of [53].

Theorem 4.2.10 ([53][2]). *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple and $\mathbb{G}$ a compact quantum group acting on it algebraically and by orientation-preserving isometries*

[2] We want to note that Goswami erroneously referred to [71] to explain the deformation of the Hopf *-algebra. Indeed, Majid uses a reality condition and Goswami a unitarity condition, which makes the theory of Majid not applicable here. We developed a new deformation of the star structure using a unitary cocycle which resulted in definitions 4.2.2 and 4.2.4.

with the representation U. Let σ be an (algebraic) unitary dual 2-cocycle on $\mathcal{O}(\mathbb{G})$. Then

(a) there exists a representation $\pi_\sigma : \mathcal{A}_{\sigma^{-1}}\#\mathbb{C} \to B(\mathcal{H})$

(b) $(\mathcal{A}_{\sigma^{-1}}\#\mathbb{C}, \mathcal{H}, D)$ is a spectral triple.

Proof. (a) For the coaction $\alpha = \mathrm{ad}_U$ of $\mathcal{O}(\mathbb{G})$ on $\mathcal{A}$, we use the Sweedler notation $\alpha(a) = a_{(0)} \otimes a_{(1)}$. Define

$$\mathcal{N} = \{(\xi_x)_x \in \oplus_x \mathcal{H}_x = \mathcal{H} \mid \xi_x \neq 0 \text{ for finitely many } x \in \mathrm{Irred}(\mathbb{G})\}.$$

Then $\mathcal{N}$ is a dense subspace of $\mathcal{H}$ such that $U(\mathcal{N}) \subset \mathcal{N} \odot \mathcal{O}(\mathbb{G})$. For $\xi \in \mathcal{N}$, we will use the notation, $U(\xi) = \xi_{(0)} \otimes \xi_{(1)}$. Then we can define, for $(a\#1) \in \mathcal{A}_{\sigma^{-1}}\#\mathbb{C}$:

$$\pi_\sigma(a\#1) : \mathcal{N} \to \mathcal{H} : \xi \mapsto a_{(0)}\xi_{(0)}\sigma^{-1}(a_{(1)}, \xi_{(1)}).$$

In section 4.3 of [53] in it is proved that $\pi_\sigma(a)$ extends to a bounded operator on $\mathcal{H}$ for all $(a\#1) \in \mathcal{A}_{\sigma^{-1}}\#\mathbb{C}$ and that $\pi_\sigma : \mathcal{A}_{\sigma^{-1}}\#\mathbb{C} \to B(\mathcal{H})$ is a well defined *-morphism.

(b) This is theorem 4.10(4) in [53].

□

4.3 Linking dimension preserving monoidal equivalences with algebraic dual 2-cocycles

In proposition 4.1.3, we proved that there is a one-to-one correspondence between dimension-preserving unitary fiber functors on a compact quantum group $\mathbb{G}$ and 2-cocycles on the dual $\hat{\mathbb{G}}$. In the following theorem 4.3.1, we will prove that there is also an equivalence between 2-cocycles on $\hat{\mathbb{G}}$ and (algebraic) dual 2-cocycles on $\mathcal{O}(\mathbb{G})$. Moreover, we will show in theorem 4.3.2 that the bi-Galois object $\mathcal{B}$ associated to the monoidal equivalence induced by the dimension-preserving unitary fiber functor, is of the form $\mathcal{B} = \mathcal{O}(\mathbb{G})\ {}_{\sigma^{-1}}\#\mathbb{C}$ with σ the associated algebraic dual 2-cocycle.

Theorem 4.3.1. *Let $\mathbb{G}$ be a compact quantum group. If Ω is a unitary 2-cocycle on the dual $\hat{\mathbb{G}}$, the formula*

$$\sigma(u^x_{ij}, u^y_{kl}) = \langle \xi^x_i \otimes \xi^y_k, \Omega(\xi^x_j \otimes \xi^y_l)\rangle, x, y \in \mathrm{Irred}(\mathbb{G}) \tag{4.3.1}$$

defines a unique (algebraic) unitary dual 2-cocycle σ on $\mathcal{O}(\mathbb{G})$. On the other hand, if σ is an (algebraic) unitary dual 2-cocycle on $\mathcal{O}(\mathbb{G})$, formula (4.3.1) *uniquely defines a unitary 2-cocycle Ω on $\hat{\mathbb{G}}$.*

Proof. Under the first assumption, as the u_{ij}^x constitute a basis of $\mathcal{O}(\mathbb{G})$, the bilinear map σ is well defined. Under the second assumption, Ω is a well defined element of $\mathcal{M}(c_0(\hat{\mathbb{G}}) \otimes c_0(\hat{\mathbb{G}}))$. Moreover, for $a = u_{ij}^x$, $b = u_{kl}^y$, $c = u_{st}^z$, we have

$$\begin{aligned}
&\sigma(a_{(1)}, b_{(1)})\sigma(a_{(2)}b_{(2)}, c) \\
&= \sum_{m,n} \sigma(u_{im}^x, u_{kn}^y)\sigma(u_{mj}^x u_{nl}^y, u_{st}^z) \\
&= \sum_{m,n} \langle \xi_i^x \otimes \xi_k^y, \Omega(\xi_m^x \otimes \xi_n^y)\rangle \\
&\quad \langle \xi_m^x \otimes \xi_n^y \otimes \xi_s^z, \big((\hat{\Delta} \otimes \mathrm{id})\Omega\big)(\xi_j^x \otimes \xi_l^y \otimes \xi_t^z)\rangle \\
&= \langle \xi_i^x \otimes \xi_k^y \otimes \xi_s^z, (\Omega \otimes \mathrm{id})\big((\hat{\Delta} \otimes \mathrm{id})\Omega\big)(\xi_j^x \otimes \xi_l^y \otimes \xi_t^z)\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\sigma(b_{(1)}, c_{(1)})\sigma(a, b_{(2)}c_{(2)}) \\
&= \sum_{m,n} \sigma(u_{km}^y, u_{sn}^z)\sigma(u_{ij}^x, u_{ml}^y u_{nt}^z) \\
&= \sum_{m,n} \langle \xi_k^y \otimes \xi_s^z, \Omega(\xi_m^y \otimes \xi_n^z)\rangle \\
&\quad \langle \xi_i^x \otimes \xi_m^y \otimes \xi_n^z, \big((\mathrm{id} \otimes \hat{\Delta})\Omega\big)(\xi_j^x \otimes \xi_l^y \otimes \xi_t^z)\rangle \\
&= \langle \xi_i^x \otimes \xi_k^y \otimes \xi_s^z, (\mathrm{id} \otimes \Omega)\big((\mathrm{id} \otimes \hat{\Delta})\Omega\big)(\xi_j^x \otimes \xi_l^y \otimes \xi_t^z)\rangle
\end{aligned}$$

which implies that σ is a dual 2-cocycle on $\mathcal{O}(\mathbb{G})$ if and only if Ω satisfies the cocycle property (4.1.1) of Ω.

Furthermore, note that as $u_{11}^{\varepsilon} = 1$, it holds that

$$\sigma(1 \otimes u_{kl}^y) = \langle \xi_1^{\varepsilon} \otimes \xi_k^y, (p_{\varepsilon} \otimes \mathrm{id})\Omega(\xi_1^{\varepsilon} \otimes \xi_l^y)\rangle$$

and

$$\sigma(u_{ij}^x \otimes 1) = \langle \xi_i^x \otimes \xi_1^{\varepsilon}, (\mathrm{id} \otimes p_{\varepsilon})\Omega(\xi_j^x \otimes \xi_1^{\varepsilon})\rangle$$

which implies that σ is normalized if and only if Ω is normalized.

Finally, denoting by σ' the dual 2-cocycle associated to Ω^*, one has

$$\begin{aligned}
\sigma'(u^x_{ij} \otimes u^y_{kl}) &= \langle \xi^x_i \otimes \xi^y_k, \Omega^*(\xi^x_j \otimes \xi^y_l)\rangle \\
&= \overline{\langle \xi^x_j \otimes \xi^y_l, \Omega(\xi^x_i \otimes \xi^y_k)\rangle} \\
&= \overline{\sigma(u^x_{ji} \otimes u^y_{lk})} \\
&= \overline{\sigma(S(u^x_{ij})^*, S(u^y_{kl})^*)}
\end{aligned}$$

Hence, if Ω is unitary, then Ω^* is the inverse of Ω and hence $\sigma' = \sigma^{-1}$ implying the unitarity condition in definition 4.2.1(3). Contrary, if the unitarity condition in definition 4.2.1(3) is satisfied, this means that $\Omega^{-1} = \Omega^*$ and hence Ω is unitary. This concludes the proof. □

Theorem 4.3.2. *Let $\mathbb{G}$ be a compact quantum group with a dimension-preserving unitary fiber functor ψ. Let $\mathcal{B}$ be the bi-Galois object associated to ψ with coaction $\beta_1 : \mathcal{B} \to \mathcal{O}(\mathbb{G}_1) \odot \mathcal{B}$, let Ω be the unitary 2-cocycle on the dual $\hat{\mathbb{G}}$ associated to $\psi \cong \psi_\Omega$ and σ the algebraic dual 2-cocycle equivalent with Ω (proposition 4.3.1). Then there exists a *-algebra isomorphism*

$$\chi : \mathcal{O}(\mathbb{G})\ {}_{\sigma^{-1}}\#\mathbb{C} \to \mathcal{B}$$

such that $(\mathrm{id}_{\mathcal{O}(\mathbb{G}_1)} \otimes \chi)\Delta = \beta_1 \circ \chi$.

Proof. Recall the monoidal equivalence $\varphi : \mathbb{G} \to \mathbb{G}_\Omega$ associated to ψ, the unitaries $u_x = \mathcal{H}_x \to \mathcal{H}_{\varphi(x)}$, the elements $Y^x = X^x(u_x \otimes 1_{\mathcal{B}}) \in B(\mathcal{H}_x) \odot \mathcal{B}$ and

$$Y' = \oplus_{x \in \mathrm{Irred}(\mathbb{G})} Y^x \in \mathcal{M}(c_0(\hat{\mathbb{G}}) \otimes B_r)$$

(where we take the direct sum over all classes, all of them with multiplicity one) which we defined in the proof of proposition 4.1.3. Now as the matrix coefficients of the X^x constitute a basis of $\mathcal{B}$ by theorem 2.6.5 and as the u_x are unitaries, also the matrix coefficients of the Y^x (let's call them b'^x_{ij}) and hence of Y' form a basis of $\mathcal{B}$. As both the $(u^x_{ij})_{ij,x}$ and $(b'^x_{ij})_{ij,x}$ are bases of $\mathcal{O}(\mathbb{G})$ resp. $\mathcal{B}$, we have a vector space isomorphism

$$\chi : \mathcal{O}(\mathbb{G}) \to \mathcal{B} : u^x_{ij} \mapsto b'^x_{ij}$$

such that $(\mathrm{id}_{\mathcal{H}_x} \otimes \chi)U^x = Y^x$ for all $x \in \mathrm{Irred}(\mathbb{G}_1)$. Moreover, χ is compatible with the coactions (i.e. $(\mathrm{id} \otimes \chi)\Delta = \beta_1 \circ \chi$). Indeed, as $(\mathrm{id} \otimes \beta_1)X^x = U^x_{12}X^x_{13}$, also $(\mathrm{id} \otimes \beta_1)Y^x = (\mathrm{id} \otimes \beta_1)(X^x(u_x \otimes 1_{\mathcal{B}})) = U^x_{12}Y^x_{13}$ and hence

$$(\mathrm{id} \otimes \beta_1 \circ \chi)U^x = (\mathrm{id} \otimes \beta_1)Y^x = U^x_{12}Y^x_{13} = (\mathrm{id} \otimes \mathrm{id} \otimes \chi)(U^x_{12}U^x_{13}) = (\mathrm{id} \otimes (\mathrm{id} \otimes \chi)\Delta)U^x$$

for all $x \in \mathrm{Irred}(\mathbb{G})$ implying $(\mathrm{id}_{\mathcal{O}(\mathbb{G}_1)} \otimes \chi)\Delta = \beta_1 \circ \chi$.

Furthermore, remind also formula (4.1.2) from the proof of proposition 4.1.3:

$$(\hat{\Delta} \otimes \mathrm{id})(Y') = Y'_{13}Y'_{23}(\Omega \otimes 1_{\mathcal{B}}).$$

As $(\hat{\Delta} \otimes \mathrm{id})(Y') = (\hat{\Delta} \otimes \chi)(\mathbb{V})$ by construction and $(\hat{\Delta} \otimes \mathrm{id})(\mathbb{V}) = (\mathbb{V}_{13}\mathbb{V}_{23})$ by definition of $\mathbb{V}$, we have

$$\begin{aligned}
&\chi(u^x_{ij}u^y_{st}) \\
&= \langle \xi^x_i \otimes \xi^y_s \otimes 1_{\mathcal{B}}, (\mathrm{id} \otimes \mathrm{id} \otimes \chi)(\mathbb{V}_{13}\mathbb{V}_{23})(\xi^x_j \otimes \xi^y_t \otimes 1_{\mathcal{B}})\rangle_{B_r} \\
&= \langle \xi^x_i \otimes \xi^y_s \otimes 1_{\mathcal{B}}, (\hat{\Delta} \otimes \mathrm{id})(Y')(\xi^x_j \otimes \xi^y_t \otimes 1_{\mathcal{B}})\rangle_{B_r} \\
&= \langle \xi^x_i \otimes \xi^y_s \otimes 1_{\mathcal{B}}, (Y'_{13}Y'_{23}(\Omega \otimes \mathrm{id}))(\xi^x_j \otimes \xi^y_t \otimes 1_{\mathcal{B}})\rangle_{B_r} \\
&= \sum_{p,q} \langle \xi^x_i \otimes \xi^y_s \otimes 1_{\mathcal{B}}, Y'_{13}Y'_{23}(\xi^x_p \otimes \xi^y_q \otimes 1_{\mathcal{B}})\rangle \langle \xi^x_p \otimes \xi^y_q, \Omega(\xi^x_j \otimes \xi^y_t)\rangle_{B_r} \\
&= \sum_{p,q} \chi(u^x_{ip})\chi(u^y_{sq})\sigma(u^x_{pj}, u^y_{qt})
\end{aligned}$$

where we used theorem 4.3.1 and equation (4.1.2) and where we used the B_r valued inproduct $\langle \cdot, \cdot \rangle_{B_r}$.
Hence, also $\chi(u^x_{ij})\chi(u^y_{st}) = \sum_{k,l} \chi(u^x_{ik}u^y_{sl})\sigma^{-1}(u^x_{kj}, u^y_{lt})$, which means

$$\chi(a)\chi(b) = \chi(a_{(0)}b_{(0)})\sigma^{-1}(a_{(1)}, b_{(1)}). \tag{4.3.2}$$

We can therefore write: $\chi : \mathcal{O}(\mathbb{G})\ {}_{\sigma^{-1}}\#\mathbb{C} \to \mathcal{B}$.

Finally, to check that χ is a *-algebra isomorphism, note that by the previous equation, we also have

$$\chi(ab^*) = \chi(a_{(0)})\chi(b^*_{(0)})\sigma(a_{(1)}, b^*_{(1)})$$

and hence

$$\begin{aligned}\chi(u^x_{ij})^* &= \sum_k \chi(u^x_{kj})^*\delta_{i,k}\\ &= \sum_{k,l}\chi(u^x_{kj})^*\chi\big(u^x_{kl}(u^x_{il})^*\big)\\ &= \sum_{k,l,p,q}\chi(u^x_{kj})^*\chi(u^x_{kp})\chi\big((u^x_{iq})^*\big)\sigma\big(u^x_{pl},(u^x_{ql})^*\big)\\ &= \sum_{l,q}\chi\big((u^x_{iq})^*\big)\sigma\big(u^x_{jl},(u^x_{ql})^*\big)\end{aligned}$$

by unitarity of the U^x and the Y^x, which implies

$$\chi(a)^* = \chi(a^*_{(1)})\sigma(S(a_{(3)})^*, a^*_{(2)}) = \chi(a^*_{(1)})V(a^*_{(2)}) \tag{4.3.3}$$

where $V(a) = \sigma(S^{-1}(a_{(2)}), a_{(1)})$ as before. This concludes the proof. $\square$

4.4 Dimension preserving monoidal deformation is isomorphic to algebraic 2-cocycle deformation

In this last section of chapter 4, we state and prove the main result of this chapter: the Goswami-Joardar cocycle deformation is a special case of our monoidal deformation with a dimension-preserving monoidal equivalence.

Theorem 4.4.1. *Let $(\mathcal{A},\mathcal{H},D)$ be a spectral triple, $\mathbb{G}$ a compact quantum group acting on it algebraically and by orientation-preserving isometries with a unitary representation U and let ψ be a dimension-preserving unitary fiber functor on $\mathbb{G}$. Denoting by $\mathcal{B}$ the corresponding bi-Galois object, there exists an (algebraic) unitary dual 2-cocycle σ such that $(\mathcal{A} \underset{\mathcal{O}(\mathbb{G})}{\boxdot} \mathcal{B}, \mathcal{H} \underset{C(\mathbb{G})}{\boxtimes} L^2(\mathcal{B}), \tilde{D})$ defined in section 3.3 and $(\mathcal{A}_{\sigma^{-1}}\#\mathbb{C},\mathcal{H},D)$ are isomorphic as spectral triples.*

Recall that $\mathcal{B}$ is the bi-Galois object associated to the fiber functor ψ, $L^2(\mathcal{B})$ the GNS-space with respect to the invariant state $\omega = (h\otimes \mathrm{id})\beta_1$ and the deformed Dirac operator $\tilde{D}$ from section 3.3. We give the proof via some propositions.

Proposition 4.4.2. *1. There exists a unitary $Y \in \mathcal{M}(\mathcal{K}(\mathcal{H})\otimes B_r) = B(\mathcal{H}\otimes B_r)$ such that $\phi : \mathcal{H} \to \mathcal{H} \underset{C(\mathbb{G})}{\boxtimes} L^2(\mathcal{B}) : \xi \to Y(\xi\otimes 1)$ is an isomorphism of Hilbert spaces.*

2. Under this isomorphism, $\phi D = \tilde{D}\phi$.

3. $\mathcal{A} \underset{\mathcal{O}(\mathbb{G})}{\boxdot} \mathcal{B} \cong \mathcal{A}_{\sigma^{-1}}\#\mathbb{C}$ with σ the algebraic dual 2-cocycle associated to the dimension-preserving unitary fiber functor ψ.

Proof. 1. Recall the unitaries $u_x : \mathcal{H}_x \to \mathcal{H}_{\varphi(x)}$, the $Y^x = X^x(u_x \otimes \mathrm{id}) \in B(\mathcal{H}_x)\odot\mathcal{B}$ from the proof of theorem 4.3.2 and the mutually inverse unitaries

$$f_x : \mathcal{H}_{\varphi(x)} \to \mathcal{H}_x \underset{C(\mathbb{G})}{\boxtimes} L^2(\mathcal{B}) : \xi \mapsto X^x(\xi \otimes \Lambda(1_{\mathcal{B}}))$$

and

$$g_x : \mathcal{H}_x \underset{C(\mathbb{G})}{\boxtimes} L^2(\mathcal{B}) \to \mathcal{H}_{\varphi(x)} : z \mapsto (\mathrm{id}_{\mathcal{H}_{\varphi(x)}} \otimes\omega_1')(X^{x*}z)$$

from the proof of proposition 3.3.2(1). Then, defining

$$\phi_x = f_x \circ u_x : \mathcal{H}_x \to \mathcal{H}_x \underset{C(\mathbb{G})}{\boxtimes} L^2(\mathcal{B}) : \xi \mapsto Y^x(\xi \otimes \Lambda(1_{\mathcal{B}}))$$

$$\phi_x' = u_x^* \circ g_x : \mathcal{H}_x \underset{C(\mathbb{G})}{\boxtimes} L^2(\mathcal{B}) \to \mathcal{H}_x : z \mapsto (\mathrm{id}_{\mathcal{H}_x} \otimes\omega_1')(Y^{x*}z),$$

obviously $\phi_x' = \phi_x^{-1}$ and defining $Y = \oplus_{x\in\mathrm{Irred}(\mathbb{G})}Y^x$, we can make $\phi = \sum_{x\in\mathrm{Irred}(\mathbb{G})} \phi_x$ (where in both cases we take the sum over the irreducible representations appearing in the decomposition of U) such that $\phi(\xi) = Y(\xi\otimes 1)$ for $\xi \in \mathcal{H}$. Y is unitary and hence ϕ is the desired isomorphism of Hilbert spaces.

2. We have to prove that, for $\xi \in \mathrm{dom}(D)$, $\phi(\xi) \in \mathrm{dom}(\tilde{D})$ and $\phi(D\xi) = \tilde{D}(\phi(\xi))$. Denote by P_λ resp. $\tilde{P}_\lambda$ the projection onto the eigenspaces V_λ resp. $V_\lambda \underset{C(\mathbb{G})}{\boxtimes} L^2(\mathcal{B})$ of D resp. $\tilde{D}$ associated to eigenvalues λ. Then note that, as $Y = (\mathrm{id}\otimes\chi)(U)$ and U commutes with D, $\phi(P_{\lambda_n}\xi) = \tilde{P}_{\lambda_n}(\phi(\xi))$. Then

$$\sum_n |\lambda_n|^2\|\tilde{P}_{\lambda_n}(\phi(\xi))\|^2 = \sum_n |\lambda_n|^2\|\phi(P_{\lambda_n}(\xi))\|^2 = \sum_n |\lambda_n|^2\|P_{\lambda_n}(\xi)\|^2 < \infty$$

for $\xi \in \mathrm{dom}(D)$ and hence ϕ maps the domain of D into the domain of $\tilde{D}$. Also, by the previous remark, trivially, $\tilde{D}_n = \tilde{D}_{|_{V_{\lambda_n} \underset{C(\mathbb{G})}{\boxtimes} L^2(\mathcal{B})}}$ commutes with ϕ for all n. Taking the direct sum, we can conclude that also $\tilde{D}$ commutes with ϕ.

3. The proof follows from theorem 4.2.9 and theorem 4.3.2.

□

Finally, it suffices to prove that the actions of the algebras on the Hilbert spaces are isomorphic.

Proposition 4.4.3. *The action of $\mathcal{A}_{\sigma^{-1}}\#\mathbb{C}$ on $\mathcal{H}$ is isomorphic to the action of $\mathcal{A} \underset{\mathcal{O}(\mathbb{G})}{\boxdot} \mathcal{B}$ on $\mathcal{H} \underset{C(\mathbb{G})}{\boxtimes} L^2(\mathcal{B})$ i.e. if $\phi : \mathcal{H} \to \mathcal{H} \underset{C(\mathbb{G})}{\boxtimes} L^2(\mathcal{B})$ and $\lambda := (\mathrm{id} \otimes \chi)\alpha_U : \mathcal{A}_{\sigma^{-1}}\#\mathbb{C} \to \mathcal{A} \underset{\mathcal{O}(\mathbb{G})}{\boxdot} \mathcal{B}$ are the isomorphisms of the previous proposition, we have:*

$$\phi(\pi_\sigma(a\#1)\xi) = \lambda(a)\phi(\xi)$$

where π_σ is as defined in theorem 4.2.10.

Proof. Let $a \in \mathcal{A}$ and let $\xi_n^{z,m}$ be the n-th basisvector in the m-th summand of $\mathcal{H}_z$ in the decomposition of $\mathcal{H}$. Using the Hilbert space isomorphism $\phi : \mathcal{H} \to \mathcal{H} \underset{C(\mathbb{G})}{\boxtimes} L^2(\mathcal{B})$, we will prove that $\phi(\pi_\sigma(a\#1)\xi_n^{z,m}) = (\mathrm{id} \otimes \chi)\alpha_U(a)\phi(\xi_n^{z,m})$ for $a\#1 \in \mathcal{A}_{\sigma^{-1}}\#\mathbb{C}$ by proving

$$\pi_\sigma(a\#1)\xi_n^{z,m} = Y^*(\mathrm{id} \otimes \chi)\alpha_U(a)(Y(\xi_n^{z,m} \otimes 1)).$$

First we compute $\pi_\sigma(a\#1)\xi_n^{z,m}$. Writing

$$U(\xi_j^{x,k} \otimes 1_{C(\mathbb{G})}) = \sum_i \xi_i^{x,k} \otimes u_{ij}^x, \tag{4.4.1}$$

it is only a calculation to check that

$$\alpha_U(a)(\xi_p^{y,l} \otimes \mathrm{id}) = U(a \otimes 1)U^*(\xi_p^{y,l} \otimes \mathrm{id}) = \sum_{x,k,i,j,q} \xi_i^{x,k}\langle \xi_j^{x,k}, a\xi_q^{y,l}\rangle \otimes u_{ij}^x (u_{pq}^y)^* \tag{4.4.2}$$

and hence that, as $\pi_\sigma(a\#1)\xi = a_{(0)}\xi_{(0)}\sigma^{-1}(a_{(1)}, \xi_{(1)})$, one has

$$\pi_\sigma(a\#1)\xi_n^{z,m} = \sum_{x,k,i,j,q} \xi_i^{x,k}\langle \xi_j^{x,k}, a\xi_q^{z,m}\rangle \sum_s \sigma^{-1}(u_{ij}^x(u_{sq}^z)^*, u_{sn}^z) \tag{4.4.3}$$

which is a finite sum as $(\alpha_U)_{|\mathcal{A}}$ is an algebraic coaction. To go further, observe that, for arbitrary elements a, b and c in $\mathcal{O}(\mathbb{G})$, we have

$$\begin{aligned}
\sigma^{-1}(ab, c) &= \sigma^{-1}(a_{(1)}b_{(1)}, c)\varepsilon(a_{(2)})\varepsilon(b_{(2)}) \\
&= \sigma^{-1}(a_{(1)}b_{(1)}, c)\sigma^{-1}(a_{(2)}, b_{(2)})\sigma(a_{(3)}, b_{(3)}) \\
&= \sigma^{-1}(a_{(1)}, b_{(1)}c_{(1)})\sigma^{-1}(b_{(2)}, c_{(2)})\sigma(a_{(2)}, b_{(3)})
\end{aligned}$$

using the cocycle relation, and applying this to our situation with $a = u^x_{ij}, b = (u^z_{sq})^*$ and $c = u^z_{sn}$, we get

$$\sum_s \sigma^{-1}(u^x_{ij}(u^z_{sq})^*, u^z_{sn}) = \sum_{s,p,t,r,v} \sigma^{-1}(u^x_{ip}, (u^z_{st})^* u^z_{sv})\sigma^{-1}((u^z_{tr})^*, u^z_{vn})\sigma(u^x_{pj}, (u^z_{rq})^*)$$

which, using the unitarity relations of the (u^z_{ij}) and $\sigma^{-1}(u^x_{ip}, 1) = \delta_{i,p}$, simplifies to

$$\sum_s \sigma^{-1}(u^x_{ij}(u^z_{sq})^*, u^z_{sn}) = \sum_{t,r} \sigma^{-1}((u^z_{tr})^*, u^z_{tn})\sigma(u^x_{ij}, (u^z_{rq})^*)$$

and hence we get

$$\pi_\sigma(a\#1)\xi^{z,m}_n = \sum_{x,k,i,j,q} \xi^{x,k}_i \langle \xi^{x,k}_j, a\xi^{z,m}_q \rangle \sum_{t,r} \sigma^{-1}((u^z_{tr})^*, u^z_{tn})\sigma(u^x_{ij}, (u^z_{rq})^*). \tag{4.4.4}$$

Next, we will compute $Y^*(\mathrm{id}\otimes\chi)\alpha_U(a)Y(\xi^{z,m}_n \otimes 1)$. Writing $Y(\xi^{x,k}_j \otimes 1_{\mathcal{B}}) = \sum_i \xi^{x,k}_i \otimes \chi(u^x_{ij})$, we have using (4.4.2),

$$\begin{aligned}
(\mathrm{id}\otimes\chi)(\alpha_U(a))Y(\xi^{z,m}_n \otimes 1) &= (\mathrm{id}\otimes\chi)\alpha_U(a)\big(\sum_t \xi^{z,m}_t \otimes \chi(u^z_{tn})\big) \\
&= \sum_{x,k,r,s,q,t} \xi^{x,k}_r \langle \xi^{x,k}_s, a\xi^{z,m}_q \rangle \otimes \chi(u^x_{rs}(u^z_{tq})^*)\chi(u^z_{tn}) \\
&= \sum_{x,k,r,s,q,t,i,j} \xi^{x,k}_r \langle \xi^{x,k}_s, a\xi^{z,m}_q \rangle \otimes \chi(u^x_{ri})\chi((u^z_{tj})^*)\chi(u^z_{tn})\sigma(u^x_{is}, (u^z_{jq})^*) \\
&= \sum_{x,k,r,s,q,t,i,j,l,p} \xi^{x,k}_r \langle \xi^{x,k}_s, a\xi^{z,m}_q \rangle \otimes \chi(u^x_{ri})\chi(u^z_{tl})^*\chi(u^z_{tn}) \\
&\qquad \sigma^{-1}((u^z_{pj})^*, u^z_{pl})\sigma(u^x_{is}, (u^z_{jq})^*) \\
&= \sum_{x,k,r,s,q,i,j,p} \xi^{x,k}_r \langle \xi^{x,k}_s, a\xi^{z,m}_q \rangle \otimes \chi(u^x_{ri})\sigma^{-1}((u^z_{pj})^*, u^z_{pn})\sigma(u^x_{is}, (u^z_{jq})^*)
\end{aligned}$$

using $\chi(ab^*) = \chi(a_{(0)})\chi(b^*_{(0)})\sigma(a_{(1)}, b^*_{(1)})$ in the third equality, $\chi(a^*) = \chi(a_{(1)})^* V^{-1}(a^*_{(2)})$ in the fourth and unitarity of $(\chi(u^{z,m}_{ij}))_{ij}$ in the last one.

Furthermore,

$$\begin{aligned}
& Y^*(\mathrm{id}\otimes\chi)\alpha_U(a)Y(\xi_n^{z,m}\otimes 1) \\
= \; & Y^*\Big(\sum_{x,k,r,s,q,i,j,p} \xi_r^{x,k}\langle\xi_s^{x,k}, a\xi_q^{z,m}\rangle\otimes\chi(u_{ri}^x)\sigma^{-1}((u_{pj}^z)^*, u_{pn}^z)\sigma(u_{is}^x,(u_{jq}^z)^*)\Big) \\
= \; & \sum_{x,k,s,q,i,j,p} \xi_i^{x,k}\langle\xi_s^{x,k}, a\xi_q^{z,m}\rangle\otimes\sigma^{-1}((u_{pj}^z)^*, u_{pn}^z)\sigma(u_{is}^x,(u_{jq}^z)^*)1_{\mathcal{B}} \qquad (4.4.5)
\end{aligned}$$

Comparing this with (4.4.4) We can conclude that

$$\phi(\pi_\sigma(a\#1)\xi) = (\mathrm{id}\otimes\chi)\alpha_U(a)\phi(\xi).$$

□

With the proof of this last proposition, we have completed the proof of theorem 4.4.1.

4.5 Conclusion

In this fourth chapter, we made a link with the work of Goswami and Joardar and proved that their deformation fits into our framework as a specific case. We proved that dimension-preserving unitary fiber functors on a compact quantum group $\mathbb{G}$ are in bijective correspondence with 2-cocycles on the dual $\hat{\mathbb{G}}$ and with algebraic dual 2-cocycles on $\mathcal{O}(\mathbb{G})$. Those algebraic dual 2-cocycles are the tools which Goswami and Joardar use in their deformation method. The main result of this chapter states that the deformed spectral triple $(\mathcal{A} \underset{\mathcal{O}(\mathbb{G})}{\boxdot} \mathcal{B}, \mathcal{H} \underset{C(\mathbb{G})}{\boxtimes} L^2(\mathcal{B}), \tilde{D})$ obtained by our construction with a dimension-preserving unitary fiber functor ψ is isomorphic with $(\mathcal{A}_{\sigma^{-1}}\#\mathbb{C}, \mathcal{H}, D)$, the deformed spectral triple obtained by Goswami and Joardar, where σ is the algebraic dual 2-cocycle associated with ψ.

Appendix

As stated in a footnote concerning theorem 4.2.10, Goswami erroneously referred to [71] for the algebraic deformation with a dual 2-cocycle of the involution. Indeed in his book, Majid defines the deformation of the involution when the dual 2-cocycle satisfies a reality condition. However, here the cocycle satisfies a unitarity condition, which we didn't find in literature in an algebraic description. Therefore in the first part of this appendix, we prove that the deformed involutions as presented in definitions 4.2.3, 4.2.5 and 4.2.8 are well defined.

As we said, in [71], Majid developed a deformation of the involution if the dual 2-cocycle satisfies a reality condition. However, we claim that the formulas (2.26) in [71] are not correct. This is easy to see: the maps U and θ there are linear, which gives a problem in the definition of the deformed involution. The mistake is being made in dualizing proposition 2.3.7, as there, a correct proof is given. In the second part of this appendix, we state the right formulas and proof they are the dual formulation of proposition 2.3.7.

Deformation of the involution with a unitary cocycle.

As the deformed Hopf-algebra and bi-comodules are well defined, we only proof that the proposed involutions are compatible.

In this part, let H be a Hopf *-algebra and σ a unitary dual 2-cocycle on H. We define the following maps:

- $U : H \to \mathbb{C} : h \mapsto \sigma(h_{(1)}, S(h_{(2)}))$,
- $V : H \to \mathbb{C} : h \mapsto U(S^{-1}(h))$.

Proposition 4.5.1. *$h^{*_\sigma} = V^{-1}(h^*_{(1)})h^*_{(2)}V(h^*_{(3)})$ is a well defined involution on H^σ.*

First we do some calculations.

Lemma 4.5.2.
- $U^{-1}(h) = \sigma^{-1}(S(h_{(1)}), h_{(2)})$
- $V^{-1}(h) = U^{-1}(S^{-1}(h)) = \sigma^{-1}(S^{-1}(h_{(2)}), h_{(1)})$
- $\overline{U(h)} = U^{-1}(S^{-2}(h^*))$
- $\overline{V(h)} = V^{-1}(h^*)$ *and* $\overline{V^{-1}(h)} = V(h^*)$
- $U(gh) = \sigma^{-1}(g_{(1)}, h_{(1)})U(g_{(2)})U(h_{(2)})\sigma^{-1}(S(h_{(3)}), S(g_{(3)}))$
- $V(gh) = \sigma^{-1}(g_{(1)}, h_{(1)})V(g_{(2)})V(h_{(2)})\sigma^{-1}(S^{-1}(h_{(3)}), S^{-1}(g_{(3)}))$

Proof.

- We have

$$\begin{aligned}
&\sigma(h_{(1)}, S(h_{(2)}))\sigma^{-1}(S(h_{(3)}), h_{(4)}) \\
&= \sigma(h_{(1)}, S(h_{(4)}))\sigma(h_{(2)}S(h_{(3)}), h_{(7)})\sigma^{-1}(S(h_{(5)}), h_{(6)}) \\
&= \sigma(S(h_{(3)}), h_{(6)})\sigma(h_{(1)}, S(h_{(2)})h_{(7)})\sigma^{-1}(S(h_{(4)}), h_{(5)}) \\
&= \sigma(h_{(1)}, S(h_{(2)})h_{(3)}) \\
&= \varepsilon(h).
\end{aligned}$$

  Analogously $\sigma^{-1}(S(h_{(1)}), h_{(2)})\sigma(h_{(3)}, S(h_{(4)}))$.

- $V(h_{(1)})U^{-1}(S^{-1}(h_{(2)})) = U(S^{-1}(h)_{(2)})U^{-1}(S^{-1}(h)_{(1)}) = \varepsilon(h)$.

- We have

$$\begin{aligned}
\overline{U(h)} &= \sigma^{-1}(S(h_{(1)})^*, S^2(h_{(2)})^*) \\
&= \sigma^{-1}(S^{-1}(h^*_{(1)}), S^{-2}(h^*_{(2)})) \\
&= U^{-1}(S^{-2}(h^*)).
\end{aligned}$$

- $\overline{V(h)} = \overline{U(S^{-1}(h))} = U^{-1}\big(S^{-2}\big(S^{-1}(h)^*\big)\big) = U^{-1}(S^{-1}(h^*)) = V^{-1}(h^*)$. Hence also $\overline{V^{-1}(h)} = V(h^*)$.

- We have

$$\begin{aligned}
U(gh) &= \sigma(g_{(1)}h_{(1)}, S(g_{(2)}h_{(2)})) \\
&= \sigma^{-1}(g_{(1)}, h_{(1)})\sigma(g_{(2)}, h_{(2)})\sigma(g_{(3)}h_{(3)}, S(g_{(4)}h_{(4)})) \\
&= \sigma^{-1}(g_{(1)}, h_{(1)})\sigma(h_{(2)}, S(g_{(4)}h_{(5)}))\sigma(g_{(2)}, h_{(3)}S(g_{(3)}h_{(4)})) \\
&= \sigma^{-1}(g_{(1)}, h_{(1)})\sigma(h_{(2)}, S(g_{(3)}h_{(3)}))U(g_{(2)}) \\
&= \sigma^{-1}(g_{(1)}, h_{(1)})U(g_{(2)})\sigma(h_{(2)}, S(h_{(3)})S(g_{(3)})) \\
&\qquad \sigma(S(h_{(4)}), S(g_{(4)}))\sigma^{-1}(S(h_{(5)}), S(g_{(5)})) \\
&= \sigma^{-1}(g_{(1)}, h_{(1)})U(g_{(2)})\sigma(h_{(3)}S(h_{(4)}), S(g_{(3)})) \\
&\qquad \sigma(h_{(2)}, S(h_{(5)}))\sigma^{-1}(S(h_{(6)}), S(g_{(4)})) \\
&= \sigma^{-1}(g_{(1)}, h_{(1)})U(g_{(2)})U(h_{(2)})\sigma^{-1}(S(h_{(3)}), S(g_{(3)})).
\end{aligned}$$

- We have

$$\begin{aligned} V(gh) &= U(S^{-1}(gh)) = U\big(S^{-1}(h)S^{-1}(g)\big) \\ &= \sigma^{-1}(g_{(1)}, h_{(1)})U(S^{-1}(g_{(2)}))U(S^{-1}(h_{(2)})) \\ &\qquad \sigma^{-1}(S^{-1}(h_{(3)}), S^{-1}(g_{(3)})) \\ &= \sigma^{-1}(g_{(1)}, h_{(1)})V(g_{(2)})V(h_{(2)})\sigma^{-1}(S^{-1}(h_{(3)}), S^{-1}(g_{(3)})). \end{aligned}$$

□

Now we are ready to proof proposition 4.5.1

Lemma 4.5.3.

- $(h^{*_\sigma})^{*_\sigma} = h$
- $g^{*_\sigma} \cdot_\sigma h^{*_\sigma} = (h \cdot_\sigma g)^{*_\sigma}$
- $\Delta_\sigma(h^{*_\sigma}) = \Delta_\sigma(h)^{*_\sigma}$
- $S_\sigma\big(S_\sigma(h^{*_\sigma})^{*_\sigma}\big) = h$

Proof.

- We have

$$\begin{aligned} (h^{*_\sigma})^{*_\sigma} &= \Big(V^{-1}(h^*_{(1)})h^*_{(2)}V(h^*_{(3)})\Big)^{*_\sigma} \\ &= \overline{V^{-1}(h^*_{(1)})}\Big(V^{-1}(h_{(2)})h_{(3)}V(h_{(4)})\Big)\overline{V(h^*_{(5)})} \\ &= V(h_{(1)})V^{-1}(h_{(2)})h_{(3)}V(h_{(4)})V^{-1}(h_{(5)}) \\ &= h \end{aligned}$$

- and

$$\begin{aligned} g^{*_\sigma} \cdot_\sigma h^{*_\sigma} &= V^{-1}(g^*_{(1)})V^{-1}(h^*_{(1)})\big(g^*_{(2)} \cdot_\sigma h^*_{(2)}\big)V(g^*_{(3)})V(h^*_{(3)}) \\ &= V^{-1}(g^*_{(1)})V^{-1}(h^*_{(1)})\sigma(g^*_{(2)}, h^*_{(2)})g^*_{(3)}h^*_{(3)} \\ &\qquad \sigma^{-1}(g^*_{(4)}, h^*_{(4)})V(g^*_{(5)})V(h^*_{(5)}) \\ &= \sigma^{-1}(S^{-1}(h^*_{(1)}), S^{-1}(g^*_{(1)}))V^{-1}(g^*_{(2)}h^*_{(2)})g^*_{(3)}h^*_{(3)} \\ &\quad V(g^*_{(4)}h^*_{(4)})\sigma(S^{-1}(h^*_{(5)}), S^{-1}(g^*_{(5)})) \end{aligned}$$

$$
\begin{aligned}
&= \Big(\sigma(h_{(1)}, g_{(1)}) h_{(2)} g_{(2)} \sigma^{-1}(h_{(3)}, g_{(3)})\Big)^{*_\sigma} \\
&= (h \cdot_\sigma g)^{*_\sigma}.
\end{aligned}
$$

- Moreover,

$$
\begin{aligned}
\Delta_\sigma(h^{*_\sigma}) &= V^{-1}(h^*_{(1)}) \Delta_\sigma(h^*_{(2)}) V(h^*_{(3)}) \\
&= V^{-1}(h^*_{(1)}) h^*_{(2)} \otimes h^*_{(3)} V(h^*_{(4)}) \\
&= V^{-1}(h^*_{(1)}) h^*_{(2)} V(h^*_{(3)}) \otimes V^{-1}(h^*_{(4)}) h^*_{(5)} V(h^*_{(6)}) \\
&= \Delta_\sigma(h)^{*_\sigma}.
\end{aligned}
$$

- Note finally that

$$
\begin{aligned}
&S_\sigma(h)^{*_\sigma} \\
&= \big(U(h_{(1)}) S(h_{(2)}) U^{-1}(h_{(3)})\big)^{*_\sigma} \\
&= \overline{U(h_{(1)})} S(h_{(2)})^{*_\sigma} \overline{U^{-1}(h_{(3)})} \\
&= \overline{U(h_{(1)})} V^{-1}\big(S(h_{(2)})^*_{(1)}\big) S(h_{(2)})^*_{(2)} V\big(S(h_{(2)})^*_{(3)}\big) \overline{U^{-1}(h_{(3)})} \\
&= \overline{U(h_{(1)}) V^{-1}(S(h_{(2)}))} S(h_{(3)})^* \overline{V(S(h_{(4)})) U^{-1}(h_{(5)})} \\
&= \overline{U(h_{(1)}) U^{-1}(h_{(2)})} S(h_{(3)})^* \overline{U(h_{(4)}) U^{-1}(h_{(5)})} \\
&= S(h)^*;
\end{aligned}
$$

hence

$$
S_\sigma\Big(S_\sigma(h)^{*_\sigma}\Big)^{*_\sigma} = S(S(h)^*)^* = h
$$

concluding the proof.

□

Proposition 4.5.4. *The involution* $(1\#h)^{*_{\mathbb{C}\#_\sigma H}} = 1\#V^{-1}(h^*_{(1)})h^*_{(2)}$ *makes* $\mathbb{C}\#_\sigma H$ *a (H^σ-H)-bi-comodule* **-algebra and the involution* $(h\#1)^{*_{H_{\sigma^{-1}}\#\mathbb{C}}} = h^*_{(1)} V(h^*_{(2)})\#1$ *makes* $H_{\sigma^{-1}}\#\mathbb{C}$ *a (H-H^σ)-bi-comodule* **-algebra.*

Proof. We give the proof for $\mathbb{C}\#_\sigma H$. The proof for $H_{\sigma^{-1}}\#\mathbb{C}$ is analogous. For notational convenience, we just write $(1\#h)^*$ for the involution applied to $1\#h$. We have

$$\begin{aligned}
((1\#h)^*)^* &= \Big(1\#V^{-1}(h^*_{(1)})h^*_{(2)}\Big)^* \\
&= 1\#\overline{V^{-1}(h^*_{(1)})}V^{-1}(h_{(2)})h_{(3)} \\
&= 1\#V(h_{(1)})V^{-1}(h_{(2)})h_{(3)} \\
&= 1\#h
\end{aligned}$$

and

$$\begin{aligned}
&(1\#g)^*(1\#h)^* \\
&\quad= V^{-1}(g^*_{(1)})V^{-1}(h^*_{(1)})\big((1\#g^*_{(2)})(1\#h^*_{(2)})\big) \\
&\quad= V^{-1}(g^*_{(1)})V^{-1}(h^*_{(1)})\sigma(g^*_{(2)},h^*_{(2)})(1\#g^*_{(3)}h^*_{(3)}) \\
&\quad= \sigma^{-1}(S^{-1}(h^*_{(1)}),S^{-1}(g^*_{(1)}))V^{-1}(g^*_{(2)}h^*_{(2)})(1\#(h_{(3)}g_{(3)})^*) \\
&\quad= \Big(\sigma(h_{(1)},g_{(1)})(1\#h_{(2)}g_{(2)})\Big)^* \\
&\quad= \big((1\#h)(1\#g)\big)^*.
\end{aligned}$$

Moreover, with $\beta_1 : \mathbb{C}\#_\sigma H \to H^\sigma \odot \mathbb{C}\#_\sigma H$ and $\beta_2 : \mathbb{C}\#_\sigma H \to \mathbb{C}\#_\sigma H \odot H$,

$$\begin{aligned}
\beta_1\big((1\#h)^*\big) &= V^{-1}(h^*_{(1)})\beta_1(1\#h^*_{(2)}) \\
&= V^{-1}(h^*_{(1)})h^*_{(2)} \otimes (1\#h^*_{(3)}) \\
&= h^{*_\sigma}_{(1)}V^{-1}(h^*_{(2)}) \otimes (1\#h^*_{(3)}) \\
&= h^{*_\sigma}_{(1)} \otimes (1\#h_{(2)})^* \\
&= \beta_1(1\#h)^*.
\end{aligned}$$

and

$$\begin{aligned}\beta_2\big((1\#h)^*\big) &= V^{-1}(h^*_{(1)})\beta_2(1\#h^*_{(2)})\\ &= V^{-1}(h^*_{(1)})(1\#h^*_{(2)})\otimes h^*_{(3)}\\ &= \beta_2(1\#h)^*.\end{aligned}$$

□

Proposition 4.5.5. *$A_{\sigma^{-1}}\#\mathbb{C}$ is a a right H^σ-comodule *-algebra with involution $(a\#1)^{*_{A_{\sigma^{-1}}\#\mathbb{C}}} = a^*_{(0)}V(a^*_{(1)})\#1$*

Proof. The proof is analogous as the proof of proposition 4.5.4. □

Deformation of the involution with a real cocycle.

Definition 4.5.6. *If H is a Hopf *-algebra, a dual 2-cocycle σ is called real if it satisfies*

$$\overline{\sigma(a,b)} = \sigma(S^2(b)^*, S^2(a)^*).$$

In that case, we also have

$$\overline{\sigma^{-1}(a,b)} = \sigma^{-1}(S^2(b)^*, S^2(a)^*).$$

Proposition 4.5.7. *Let H be a Hopf *-algebra and σ a real dual 2-cocycle on H. Define the following maps:*

- $U : H \to \mathbb{C} : h \mapsto \sigma(h_{(1)}, S(h_{(2)}))$,
- $W : H \to \mathbb{C} : h \mapsto U(S^{-2}(h))$,
- $V : H \to \mathbb{C} : h \mapsto W(S(h_{(1)}))W^{-1}(h_{(2)})$.

Then $h^{*_\sigma} = V^{-1}(h^*_{(1)})h^*_{(2)}V(h^*_{(3)})$ is a well defined involution on H^σ.

We will give the proof using different lemmas:

Lemma 4.5.8.
- $U^{-1}(h) = \sigma^{-1}(S(h_{(1)}), h_{(2)})$
- $W^{-1}(h) = \sigma^{-1}(S^{-1}(h_{(1)}), S^{-2}(h_{(2)}))$
- $V^{-1}(h) = W(h_{(1)})W^{-1}(S(h_{(2)}))$.

Proof.
- this is proven in lemma 4.5.2.
- $W^{-1}(h) = U^{-1}(S^{-2}(h)) = \sigma^{-1}(S^{-1}(h_{(1)}), S^{-2}(h_{(2)}))$
- $W(h_{(1)})W^{-1}(S(h_{(2)}))W(S(h_{(3)}))W^{-1}(h_{(4)}) = \varepsilon(h)$ and $W(S(h_{(1)}))W^{-1}(h_{(2)})W(h_{(3)})W^{-1}(S(h_{(4)})) = \varepsilon(h)$.

□

Lemma 4.5.9.
- $\overline{U(h)} = U(S^{-3}(h^*))$
- $\overline{W(h)} = W(S(h^*))$
- $\overline{V(h)} = W(h^*_{(1)})W^{-1}(S(h^*_{(2)})) = V^{-1}(h^*)$.

Proof.
- We have

$$\begin{aligned}\overline{U(h)} &= \overline{\sigma(h_{(1)}, S(h_{(2)}))} \\ &= \sigma(S^3(h_{(2)})^*, S^2(h_{(1)})^*) \\ &= \sigma(S^{-3}(h^*)_{(1)}, S(S^{-3}(h^*)_{(2)})) \\ &= U(S^{-3}(h^*)),\end{aligned}$$

- $\overline{W(h)} = \overline{U(S^{-2}(h))} = U(S^{-3}(S^{-2}(h)^*)) = U(S^{-3}(S^2(h^*))) = W(S(h^*))$,
- and

$$\begin{aligned}\overline{V(h)} &= \overline{W(S(h_{(1)}))W^{-1}(h_{(2)})} \\ &= W(S(S(h_{(1)})^*))W^{-1}(S(h^*_{(2)})) \\ &= W(h^*_{(1)})W^{-1}(S(h^*_{(2)})) \\ &= V^{-1}(h^*).\end{aligned}$$

□

Lemma 4.5.10.
- $U(gh) = \sigma^{-1}(g_{(1)}, h_{(1)})U(g_{(2)})U(h_{(2)})\sigma^{-1}(S(h_{(3)}), S(g_{(3)}))$
- $W(gh) = \sigma^{-1}(S^{-2}(g_{(1)}), S^{-2}(h_{(1)}))W(g_{(2)})W(h_{(2)})\sigma^{-1}(S^{-1}(h_{(3)}), S^{-1}(g_{(3)}))$
- $V(gh) = \sigma^{-1}(g_{(1)}, h_{(1)})V(g_{(2)})V(h_{(2)})\sigma(S^{-2}(g_{(3)}), S^{-2}(h_{(3)}))$

Proof.

- This is proven in lemma 4.5.2.
- We have

$$
\begin{aligned}
& W(gh) \\
&= U(S^{-2}(gh)) = U(S^{-2}(g)S^{-2}(h)) \\
&= \sigma^{-1}(S^{-2}(g_{(1)}), S^{-2}(h_{(1)}))W(g_{(2)})W(h_{(2)}) \\
&\qquad\qquad \sigma^{-1}(S^{-1}(h_{(3)}), S^{-1}(g_{(3)}))
\end{aligned}
$$

- and

$$
\begin{aligned}
& V(gh) \\
&= W\big(S(g_{(1)}h_{(1)})\big)W^{-1}(g_{(2)}h_{(2)}) \\
&= W\big(S(h_{(1)})S(g_{(1)})\big)W^{-1}\big(g_{(2)}h_{(2)}\big) \\
&= \sigma^{-1}(g_{(1)}, h_{(1)})W(S(g_{(2)}))W(S(h_{(2)})) \\
&\qquad \sigma^{-1}(S^{-1}(h_{(3)}), S^{-1}(g_{(3)}))\sigma(S^{-1}(h_{(4)}), S^{-1}(g_{(4)})) \\
&\qquad W^{-1}(g_{(5)})W^{-1}(h_{(5)})\sigma(S^{-2}(g_{(6)}), S^{-2}(h_{(6)})) \\
&= \sigma^{-1}(g_{(1)}, h_{(1)})V(g_{(2)})V(h_{(2)})\sigma(S^{-2}(g_{(3)}), S^{-2}(h_{(3)})).
\end{aligned}
$$

□

Proposition 4.5.11. *We have*

- $(h^{*_\sigma})^{*_\sigma} = h$,
- $(g \cdot_\sigma h)^{*_\sigma} = h^{*_\sigma} \cdot_\sigma g^{*_\sigma}$,
- $\Delta(h^{*_\sigma}) = (h_{(1)})^{*_\sigma} \otimes (h_{(2)})^{*_\sigma}$,
- $S_\sigma\big(S_\sigma(h^{*_\sigma})^{*_\sigma}\big) = h$.

Proof. • We have

$$\begin{aligned}
(h^{*_\sigma})^{*_\sigma} &= \Big(V^{-1}(h^*_{(1)})h^*_{(2)}V(h^*_{(3)})\Big)^{*_\sigma} \\
&= \overline{V^{-1}(h^*_{(1)})}V^{-1}(h_{(2)})h_{(3)}V(h_{(4)})\overline{V(h^*_{(5)})} \\
&= V(h_{(1)})V^{-1}(h_{(2)})h_{(3)}V(h_{(4)})V^{-1}(h_{(5)}) \\
&= h
\end{aligned}$$

• We have

$$\begin{aligned}
& h^{*_\sigma} \cdot_\sigma g^{*_\sigma} \\
&= \big(V^{-1}(h^*_{(1)})h^*_{(2)}V(h^*_{(3)})\big) \cdot_\sigma \big(V^{-1}(g^*_{(1)})g^*_{(2)}V(g^*_{(3)})\big) \\
&= V^{-1}(h^*_{(1)})V^{-1}(g^*_{(1)})\sigma(h^*_{(2)}, g^*_{(2)})h^*_{(3)}g^*_{(3)} \\
&\qquad\qquad \sigma^{-1}(h^*_{(4)}, g^*_{(4)})V(h^*_{(5)})V(g^*_{(5)}) \\
&= \sigma(S^{-2}(h^*_{(1)}), S^{-2}(g^*_{(1)}))V^{-1}(h^*_{(2)}g^*_{(2)})h^*_{(3)}g^*_{(3)}V(h^*_{(4)}g^*_{(4)}) \\
&\qquad\qquad \sigma^{-1}(S^{-2}(h^*_{(5)}), S^{-2}(g^*_{(5)})) \\
&= \overline{\sigma(g_{(1)}, h_{(1)})}(g_{(2)}h_{(2)})^{*_\sigma}\overline{\sigma^{-1}(g_{(3)}, h_{(3)})} \\
&= \Big(\sigma(g_{(1)}, h_{(1)})g_{(2)}h_{(2)}\sigma^{-1}(g_{(3)}, h_{(3)})\Big)^{*_\sigma} \\
&= (g \cdot_\sigma h)^{*_\sigma}
\end{aligned}$$

• We have

$$\begin{aligned}
\Delta(h^{*_\sigma}) &= \Delta(V^{-1}(h^*_{(1)})h^*_{(2)}V^*(h_{(3)})) \\
&= V^{-1}(h^*_{(1)})h^*_{(2)} \otimes h^*_{(3)}V(h^*_{(4)}) \\
&= V^{-1}(h^*_{(1)})h^*_{(2)}V(h^*_{(3)}) \otimes V^{-1}(h^*_{(4)})h^*_{(5)}V(h^*_{(6)}) \\
&= (h_{(1)})^{*_\sigma} \otimes (h_{(2)})^{*_\sigma}
\end{aligned}$$

- Note finally that

$$
\begin{aligned}
& S_\sigma(h)^{*_\sigma} \\
&= \big(U(h_{(1)})S(h_{(2)})U^{-1}(h_{(3)})\big)^{*_\sigma} \\
&= \overline{U(h_{(1)})}S(h_{(2)})^{*_\sigma}\overline{U^{-1}(h_{(3)})} \\
&= \overline{U(h_{(1)})}V^{-1}\big(S(h_{(2)})^*_{(1)}\big)S(h_{(2)})^*_{(2)}V\big(S(h_{(2)})^*_{(3)}\big)\overline{U^{-1}(h_{(3)})} \\
&= \overline{U(h_{(1)})V^{-1}(S(h_{(2)}))}S(h_{(3)})^*\overline{V(S(h_{(4)}))U^{-1}(h_{(5)})} \\
&= \overline{U(h_{(1)})W^{-1}(S^2(h_{(2)}))W(S(h_{(3)}))}S(h_{(4)})^* \\
&\qquad \overline{W^{-1}(S(h_{(5)}))W(S^2(h_{(6)}))U^{-1}(h_{(7)})} \\
&= \overline{U(h_{(1)})U^{-1}(h_{(2)})W(S(h_{(3)}))}S(h_{(4)})^* \\
&\qquad \overline{W^{-1}(S(h_{(5)}))U(h_{(6)})U^{-1}(h_{(7)})} \\
&= W(h^*_{(1)})S(h_{(2)})^*W^{-1}(h^*_{(3)})
\end{aligned} \tag{4.5.1}
$$

hence

$$
\begin{aligned}
& S_\sigma\Big(S_\sigma(h)^{*_\sigma}\Big)^{*_\sigma} \\
&= S_\sigma\Big(W(h^*_{(1)})S(h_{(2)})^*W^{-1}(h^*_{(3)})\Big)^{*_\sigma} \\
&= \Big(W(h^*_{(1)})S_\sigma\big(S(h_{(2)})^*\big)W^{-1}(h^*_{(3)})\Big)^{*_\sigma} \\
&= \overline{W(h^*_{(1)})}S_\sigma\Big(S(h_{(2)})^*\Big)^{*_\sigma}\overline{W(^{-1}h^*_{(3)})} \\
&= \overline{W(h^*_{(1)})}W(S(h_{(2)})_{(1)})S\big(S(h_{(2)})^*_{(2)}\big)^*W^{-1}(S(h_{(2)})_{(3)})\overline{W^{-1}(h^*_{(3)})} \\
&= W(S(h_{(1)}))W^{-1}(S(h_{(2)}))h_{(3)}W(S(h_{(4)}))W^{-1}(S(h_{(5)})) \\
&= h
\end{aligned}
$$

concluding the proof.

□

Finally, we proof that this involution is indeed the dual of the deformed involution ${}^{*_\chi} = (S^{-1}U')((\cdot))^*(S^{-1}U'^{-1})$, defined in proposition 2.3.7 of [71] where U' is the dual of U.

Proposition 4.5.12. *The new defined involution ${}^{*_\sigma}$ is the dual of the deformed involution ${}^{\star_\sigma} = (S^{-1}U)((\cdot))^*(S^{-1}U^{-1})$, defined in proposition 2.3.7 of [71].*

Proof. Let H^* be the dual of the Hopf algebra H with duality relation $H^* \odot H \to \mathbb{C} : a \otimes h \mapsto \langle a, h \rangle$. For notational convenience, we denote by $\chi \in H^* \odot H^*$ the 2-cocycle on H^* such that $(H^\sigma)^*$ is isomorphic to H^* as algebra and with coalgebra structure $\Delta_\chi(a) = \chi\Delta(a)\chi^{-1}$ and antipode $S_\chi(a) = U'S(a)U'^{-1}$ where $U' = \chi^1 S(\chi^2)$ and $\chi = \chi^1 \otimes \chi^2$. To prove that ${}^{*_\sigma}$ is the dual of the deformed involution ${}^{\star_\sigma} ={}^{*_\chi} (S^{-1}U)((\cdot))^*(S^{-1}U^{-1})$, we have to check that

$$\langle a^{*_\chi}, h \rangle = \overline{\langle a, (S_\sigma(h))^{*_\sigma} \rangle}.$$

We have

$$\begin{aligned}
\overline{\langle a, (S_\sigma(h))^{*_\sigma} \rangle} &= \overline{\langle a, W(h^*_{(1)})S(h_{(2)})^* W^{-1}(h^*_{(3)}) \rangle} \\
&= W(S(h_{(1)}))\overline{\langle a, S(h_{(2)})^* \rangle} W^{-1}(S(h_{(3)})) \\
&= U(S^{-1}(h_{(1)}))\overline{\langle a, S(h_{(2)})^* \rangle} U^{-1}(S^{-1}(h_{(3)})) \\
&= (S^{-1}U')(h_{(1)})\langle a^*, h_{(2)} \rangle (S^{-1}U'^{-1})(h_{(3)}) \\
&= \langle (S^{-1}U') \otimes a^* \otimes (S^{-1}U'^{-1}), h_{(1)} \otimes h_{(2)} \otimes h_{(3)} \rangle \\
&= \langle (S^{-1}U')(a)^*(S^{-1}U'^{-1}), h \rangle \\
&= \langle a^{*_\chi}, h \rangle
\end{aligned}$$

where we used (4.5.1) of proposition 4.5.11. This completes the proof. □

Chapter 5

Constructing a non-dimension-preserving example

In chapter 3 we proposed a new method to deform spectral triples which we called monoidal deformation. In the previous chapter we proved that the 2-cocycle deformation introduced by Goswami and Joardar in [53] fits into the framework of monoidal deformation as a special case. In this fifth chapter we prove that this method is a proper generalization by constructing an example of a monoidal deformation coming from a non-dimension-preserving monoidal equivalence. We will use the spectral triple on the Podleś spheres ([78]) defined in [36] and $SU_q(2)$, which acts on it algebraically and by orientation preserving isometries. The chapter is structured as follows. In the first section we investigate unitary fiber functors on $SU_q(2)$ and in the last section, we construct the example.

5.1 Monoidal equivalences on $SU_q(2)$

In this first section we look at orthogonal quantum groups and $SU_q(2)$ in particular. Moreover, we investigate their monoidal equivalences.

Definition 5.1.1 ([94]). *Let $n \in \mathbb{N}$ and $F \in \mathrm{GL}(n, \mathbb{C})$ with $F\overline{F} = cI_n \in \mathbb{R}I_n$. Then $A_o(F)$ is defined as the universal quantum group generated by the coefficients of the matrix $U \in M_n(A_o(F))$ with relations*

- *U is a unitary and*
- $U = F\overline{U}F^{-1}$

where $(\overline{U})_{ij} = (U_{ij})^*$. Moreover, $A_o(F) = (C(A_o(F)), U)$ is a compact matrix quantum group (as defined in [102]). They are called universal orthogonal quantum groups.

As the matrices F are not in one-to-one correspondence with the universal quantum groups (i.e. different F's can define the same universal quantum group), it is necessary (but not so hard) to classify the quantum groups $A_o(F)$. This has been done in [27].

Proposition 5.1.2 ([27])**.** *For F_1, F_2 matrices in $\mathrm{GL}(n, \mathbb{C})$ with $F_i\overline{F}_i = \pm 1$, we say*

$$F_1 \sim F_2 \text{ if there exists a unitary } v \in U(n) \text{ such that } F_1 = vF_2v^T.$$

Then

$$A_o(F_1) \cong A_o(F_2) \text{ if and only if } F_1 \sim F_2.$$

Therefore, we will describe a fundamental domain for $\sim$ as is done in [27].

Proposition 5.1.3. *A fundamental domain of $\sim$ is given by the following classes of matrices:*

- $\begin{pmatrix} 0 & D(\lambda_1, \ldots, \lambda_k) & 0 \\ D(\lambda_1, \ldots, \lambda_k)^{-1} & 0 & 0 \\ 0 & 0 & 1_{n-2k} \end{pmatrix}$ *with* $k, n \in \mathbb{N}, 2k \leq n, 0 < \lambda_1 \leq \ldots \leq \lambda_k < 1$,
- $\begin{pmatrix} 0 & D(\lambda_1, \ldots, \lambda_{n/2}) \\ -D(\lambda_1, \ldots, \lambda_{n/2})^{-1} & 0 \end{pmatrix}$ *with* $0 < \lambda_1 \leq \ldots \leq \lambda_{n/2} \leq 1, n \in \mathbb{N}$ *even,*

where $D(\lambda_1, \ldots, \lambda_i)$ is the diagonal matrix with entries $\lambda_1, \ldots, \lambda_i$ on the diagonal.

Remark 5.1.4. *Note that for $F \in \mathrm{GL}(2, \mathbb{C})$, up to equivalence, there only exists matrices of the form*

$$F_q = \begin{pmatrix} 0 & |q|^{1/2} \\ -\operatorname{sgn}(q)|q|^{-1/2} & 0 \end{pmatrix}$$

for $q \in [-1, 1] \setminus \{0\}$.

Now we have a look at deformations of $SU(2)$ and make a link with orthogonal quantum groups.

Definition 5.1.5 ([102, 103]). *Let $q \in [-1,1], q \neq 0$. Let A be the universal unital C^*-algebra generated by two elements α, γ satisfying the relations such that $U = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \in M_2(A)$ is a unitary matrix. With coproduct $\Delta(U_{ij}) = \sum_k U_{ik} \otimes U_{kj}$, $SU_q(2) = (A, \Delta)$ is a compact quantum group.*

One can easily check that for $q = 1$, $SU_q(2) = SU(2)$.

Proposition 5.1.6. *With F_q defined in remark 5.1.4, we have $A_o(F_q) \cong SU_q(2)$.*

Note that this last statement indeed implies that the only orthogonal quantum groups coming from matrices of dimension 2, are the quantized versions of $SU(2)$. Now we have a look at the monoidal equivalences of $SU_q(2)$.

We state some results obtained by Bichon et al. in [27] (Corollary 5.4 and Theorem 5.5).

Theorem 5.1.7 ([27][1]). *Let $F_1 \in GL(n_1, \mathbb{C})$ with $F_1\overline{F}_1 = c_1 1$, $c_1 \in \mathbb{R}$. Then*

- *a compact quantum group $\mathbb{G}$ is monoidally equivalent with $A_o(F_1)$ if and only if there exist a $F_2 \in GL(n_2, \mathbb{C})$ with $F_2\overline{F}_2 = c_2 1$, $c_2 \in \mathbb{R}$ and $\frac{c_1}{\mathrm{Tr}(F_1^* F_1)} = \frac{c_2}{\mathrm{Tr}(F_2^* F_2)}$ such that $\mathbb{G} \cong A_o(F_2)$.*
- *In this case, denote by $\mathcal{O}(A_o(F_1, F_2))$ the *-algebra generated by the coefficients of $Y \in M_{n_1,n_2}(\mathbb{C}) \otimes \mathcal{O}(A_o(F_1, F_2))$ with relations*

$$Y \text{ is unitary} \quad \text{and} \quad Y = (F_1 \otimes 1)\overline{Y}(F_2^{-1} \otimes 1),$$

then $\mathcal{O}(A_o(F_1, F_2)) \neq \{0\}$ and it is the ($A_o(F_1)$-$A_o(F_2)$)-bi-Galois object with left coaction β_1 of $\mathcal{O}(A_o(F_1))$ and right coaction β_2 of $\mathcal{O}(A_o(F_2))$ such that

$$(\mathrm{id} \otimes \beta_1)(Y) = (U_1)_{12} Y_{13} \quad \text{and} \quad (\mathrm{id} \otimes \beta_2)(Y) = Y_{12}(U_2)_{13}$$

where the U_i are the unitary representations of $A_o(F_i)$, whose matrix coefficients generate the quantum groups.

[1] To have a left coaction of $\mathcal{O}(A_o(F_1))$ and a right coaction of $\mathcal{O}(A_o(F_2))$ on $\mathcal{O}(A_o(F_1, F_2))$, we have to change the left-right conventions of [27].

- *The monoidal equivalence $\varphi : A_o(F_1) \to A_o(F_2)$ preserves the dimensions if and only if $n_2 = n_1$. In this case, we denote the unitary 2-cocycle associated to φ by $\Omega(F_2)$. Denoting $\Omega(F)$ to be equivalent with $\Omega(F')$ if and only if $F \sim F'$ in the sense of proposition 5.1.2, the set $\{\Omega(F_2) \mid \dim(F_2) = n_1\}$ describe up to equivalence all unitary 2-cocycles on the dual of $A_o(F_1)$.*

Remark 5.1.8. *In [2] Banica shows that the irreducible representations of $A_o(F)$ can be labeled by $\mathbb{N}$ (say r_k, $k \in \mathbb{N}$). Moreover, for $\dim(F) = n$, he states that $\dim(r_k) = (x^{k+1} - y^{k+1})/(x - y)$ where x and y are solutions of $X^2 - nX + 1 = 0$ for $n \geq 3$ and $\dim(r_k) = k + 1$ for $n = 2$. Hence, it is easy to show by induction that if φ is a monoidal equivalence between $SU_q(2)$ and $A_o(F)$ with $\dim(F) \geq 3$, then $\dim(\varphi(r_k)) > \dim(r_k) = k + 1$ for every irreducible representation r_k with $k \geq 1$.*

Proof. Let $N \geq 3$ and x and y the two solutions of $X^2 - nX + 1 = 0$. Then $xy = 1$ and $x + y = n$. Hence, suppose that $x > y$, then $x = (n + \sqrt{n^2 - 4})/2 > n - 1 \geq 2$. For $k = 1$, we have $(x^2 - y^2)/(x - y) = x + y = n \geq 3 > 2$ and hence the statement is clearly true for $k = 2$. Suppose now $k \geq 1$ and suppose that the statement is true for k (i.e. $\frac{x^{k+1} - y^{k+1}}{x - y} > k + 1$). Then we have

$$\begin{aligned}
\frac{x^{k+2} - y^{k+2}}{x - y} &= x\frac{x^{k+1} - y^{k+1}}{x - y} + \frac{xy^{k+1} - y^{k+2}}{x - y} \\
&> x(k + 1) + y^{k+1} > 2(k + 1) + y^{k+1} > k + 2.
\end{aligned}$$

This completes the proof. $\square$

Moreover, looking at the concrete orthogonal quantum group $SU_q(2)$, it is possible to classify all compact quantum groups which are monoidally equivalent with $SU_q(2)$: indeed applying theorem 5.1.7 to the specific situation $F = F_q$, we know exactly what the quantum groups are which are monoidally equivalent with $SU_q(2)$.

Proposition 5.1.9 ([27]). *Let $0 < q \leq 1$. For every even natural number n with $2 \leq n \leq q + 1/q$, there exists a monoidal equivalence on $SU_q(2)$ such that the multiplicity of the fundamental representation is n. Concretely, $SU_q(2) \sim_{mon} A_o(F)$ with $F = \begin{pmatrix} 0 & D(\lambda_1, \ldots, \lambda_{n/2}) \\ -D(\lambda_1, \ldots, \lambda_{n/2})^{-1} & 0 \end{pmatrix}$ where $0 < \lambda_1 \leq \ldots \leq \lambda_{n/2} \leq 1$ and $\sum_{i=1}^{n/2} \frac{1}{\lambda_i^2} + \lambda_i^2 = q + 1/q$.*

Let $0 > q \geq -1$. Then for every natural number n with $2 \leq n \leq |q + 1/q|$, *there exists a monoidal equivalence on $SU_q(2)$ such that the multiplicity of the fundamental representation is n. Concretely, $SU_q(2) \sim_{mon} A_o(F)$ with $F = \begin{pmatrix} 0 & D(\lambda_1, \dots, \lambda_k) & 0 \\ D(\lambda_1, \dots, \lambda_k)^{-1} & 0 & 0 \\ 0 & 0 & 1_{n-2k} \end{pmatrix}$ where $k \in \mathbb{N}, 2k \leq n, 0 < \lambda_1 \leq \dots \leq \lambda_k < 1$ and $\sum_{i=1}^k \frac{1}{\lambda_i^2} + \lambda_i^2 + n - 2k = |q + 1/q|$.*

5.2 Monoidal deformation of the Podleś sphere

In chapter 4, we proved that our monoidal deformation of spectral triples is a generalization of the cocycle deformation, developed in [53]. In this section, we will give a concrete example to prove that our construction is a proper generalization: we will construct a monoidal deformation of the Podleś sphere (with spectral triple of Dabrowski, Landi, Wagner and D'Andrea [36]) which is not a 2-cocycle deformation. First we recapitulate the definition of the Podleś sphere $S^2_{q,c}$ and the spectral triple on it. Then we will use the results of section 5.1 to apply the construction of section 3.3.

5.2.1 The Podleś sphere, its spectral triple and its quantum isometry group

The Podleś sphere was initially constructed by Podleś in [79] as follows. Let $q \in (-1, 1) \setminus \{0\}$ and $t \in (0, 1]$, hence $c = t^{-1} - t > 0$. We define $\mathcal{O}(S^2_{q,c})$ to be the *-algebra generated by elements A, B which satisfy the relations

$$A^* = A, \qquad AB = q^{-2}BA,$$

$$B^*B = A - A^2 + c1, \qquad BB^* = q^2A - q^4A^2 + c1.$$

One can see that for $q = 1$, we have $A^* = A$, $AB = BA$, $B^*B = BB^* = A - A^2 + c1$ and this is the classical sphere: putting $A = z + 1/2$, $B = x + iy$, $r^2 = c + 1/4$, we indeed have

$$x^2 + y^2 + z^2 = B^*B + A^2 - A + 1/4 = c + 1/4 = r^2.$$

The associated quantum space is called the Podleś sphere $S^2_{q,c}$.

Note first that for $q \in (0,1)$, setting

$$x_0 = t(1-(1+q^2)A), x_{-1} = \frac{t(1+q^2)^{\frac{1}{2}}}{q}B, x_1 = -t(1+q^2)^{\frac{1}{2}}B^*,$$

we see that the definition in [36] with $\{x_0, x_{-1}, x_1\}$ is equivalent to the original definition of Podleś given above. Moreover, defining

$$\tilde{A} = \frac{1 + t^{-1}q\gamma^*\alpha - t^{-1}\rho(1-(1+q^2)\gamma^*\gamma) + t^{-1}\gamma\alpha^*}{1+q^2}$$

$$\tilde{B} = \frac{q\alpha^2 + \rho(1+q^2)\alpha\gamma - q^2\gamma^2}{t(1+q^2)},$$

(where $\rho^2 = \frac{q^2t^2}{(q^2+1)^2(1-t)}$) if $t \neq 1$ and

$$\tilde{A} = \gamma^*\gamma, \qquad \tilde{B} = q\alpha\gamma,$$

if $t = 1$ one can prove that the unital *-subalgebra of $C(SU_q(2))$ generated by $\tilde{A}$ and $\tilde{B}$ is isomorphic to $\mathcal{O}(S^2_{q,c})$ where $c = t^{-1} - t$, sending A to $\tilde{A}$ and B to $\tilde{B}$. Doing as above, we have 3 equivalent descriptions of the Podleś sphere.

The spectral triple on $S^2_{q,c}$ we will use, is the spectral triple developed by Dabrowski, D'Andrea, Landi and Wagner in [36]. The spectral triple uses the representation theory of $SU_q(2)$ described by Banica in [2]. To be compatible with [36], we use their notation. For each n in $\{0, 1/2, 1, \ldots\}$, there exists a unique irreducible representation D^n of $SU_q(2)$ (r_{2n} in Banica's notation) with dimension $2n+1$. For example , we have

$$D^{1/2} = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

and

$$D^1 = \begin{pmatrix} \alpha^{*2} & -(q^2+1)\alpha^*\gamma & -q\gamma^2 \\ \gamma^*\alpha^* & 1-(q^2+1)\gamma^*\gamma & \alpha\gamma \\ -q\gamma^{*2} & -(q^2+1)\gamma^*\alpha & \alpha^2 \end{pmatrix}.$$

Denoting $d^n_{k,l}$ to be the k,l-matrix coefficient of D^n, one can prove that

$$\{d^n_{k,l} \mid n = 0, \frac{1}{2}, 1, \ldots; k, l = -n, -n+1, \ldots, n-1, n\}$$

form an orthogonal basis of $\mathcal{K} = L^2(SU_q(2), h)$, the GNS-space corresponding to the Haar state h of $SU_q(2)$. Moreover we will denote $e^n_{k,l}$ the multiples of $d^n_{k,l}$ such that the $\{e^n_{k,l}\}$ form an orthonormal basis of $\mathcal{K}$.

Furthermore, consider the following closed subspace of $\mathcal{K}$

$$\mathcal{H} := \left[e^n_{\pm\frac{1}{2},l} \mid n = \frac{1}{2}, \frac{3}{2}, \dots ; l = -n, -n+1, \dots, n-1, n\right].$$

Then one can prove that $\tilde{A}$ and $\tilde{B}$, as defined above, leave $\mathcal{H}$ invariant and we have a faithful *-morphism $\pi : \mathcal{O}(S^2_{q,c}) \to B(\mathcal{H}) : A \mapsto \tilde{A}_{|_{\mathcal{H}}}, B \mapsto \tilde{B}_{|_{\mathcal{H}}}$, which makes it possible to identify $\mathcal{O}(S^2_{q,c})$ with its image.

Finally, we can define an appropriate Dirac operator by setting

$$D(e^n_{\pm\frac{1}{2},l}) = (c_1 n + c_2) e^n_{\mp\frac{1}{2},l}$$

where $c_1, c_2 \in \mathbb{R}$, $c_1 \neq 0$ are arbitrary constants.
In [36] the authors prove that $(\mathcal{O}(S^2_{q,c}), \mathcal{H}, D)$ constitutes a well defined spectral triple. As

$$\Delta_{SU_q(2)}(e^n_{\pm\frac{1}{2},l}) = \sum_{k=-n,-n+1,\dots,n} e^n_{\pm\frac{1}{2},k} \otimes e^n_{k,l}$$

it is easy to see that $\Delta_{SU_q(2)}$ induces a unitary representation U of $SU_q(2)$ on $\mathcal{H}$. By [36] the spectral triple is equivariant with respect to this representation and hence, $SU_q(2)$ acts algebraically and by orientation-preserving isometries on $(\mathcal{O}(S^2_{q,c}), \mathcal{H}, D)$. We will use this representation and the monoidal equivalences of section 5.1 to deform this spectral triple.

5.2.2 Monoidal deformation of the Podleś sphere

To conclude this chapter, we construct the non-dimension-preserving example announced before. Now we know that there is a well defined spectral triple $(\mathcal{O}(S^2_{q,c}), \mathcal{H}, D)$ on which $SU_q(2)$ acts algebraically and by orientation-preserving isometries. Furthermore, we know from proposition 5.1.9 what the monoidal equivalences of $SU_q(2)$ are and we know that those monoidal equivalences are non-dimension-preserving by remark 5.1.8. Putting all this together, we can apply the construction described in section 3.3 to get the following theorem.

Theorem 5.2.1. *Let $q \in (-1,1) \setminus \{0\}$ and n a natural number with $3 \leq n \leq |q + 1/q|$.*
If $q > 0$ and n is even, let $\lambda_1, \dots, \lambda_{n/2}$ be real numbers with $0 < \lambda_1 \leq \dots \leq \lambda_{n/2} \leq 1$ such that $\lambda_1^2 + \dots + \lambda_{n/2}^2 + 1/\lambda_1^2 + \dots + 1/\lambda_{n/2}^2 = q + 1/q$ and define F to be the n by n matrix

$$F = \begin{pmatrix} 0 & D(\lambda_1, \dots, \lambda_{n/2}) \\ -D(\lambda_1, \dots, \lambda_{n/2})^{-1} & 0 \end{pmatrix}.$$

If $0 > q$*, let* k *be a natural number* $k \leq n/2$ *and* $\lambda_1, \dots, \lambda_k$ *be strict positive real numbers such that* $0 < \lambda_1 \leq \dots \leq \lambda_k < 1$ *and* $\sum_{i=1}^{k} \frac{1}{\lambda_i^2} + \lambda_i^2 + n - 2k = |q + 1/q|$ *and define* F *to be the* n *by* n *matrix*

$$F = \begin{pmatrix} 0 & D(\lambda_1, \dots, \lambda_k) & 0 \\ D(\lambda_1, \dots, \lambda_k)^{-1} & 0 & 0 \\ 0 & 0 & 1_{n-2k} \end{pmatrix}.$$

With F *defined as above, there exists a non-dimension-preserving monoidal equivalence* φ *from* $SU_q(2)$ *to* $A_o(F)$ *(introduced in definition 5.1.1). Denoting by* $\mathcal{O}(A_o(F_q, F))$ *the algebra constructed in 5.1.7,* $\mathcal{O}(A_o(F_q, F))$ *is the associated bi-Galois object and the following triplet is a spectral triple:*

$$\big(\mathcal{O}(S^2_{q,c}) \underset{\mathcal{O}(SU_q(2))}{\boxdot} \mathcal{O}(A_o(F_q, F)), \quad \mathcal{H} \underset{C(SU_q(2))}{\boxtimes} L^2\big(\mathcal{O}(A_o(F_q, F))\big), \quad \tilde{D}\big).$$

Moreover $A_o(F)$ *acts algebraically and by orientation-preserving isometries on the new spectral triple. As* φ *is non-dimension-preserving, it is not a 2-cocycle deformation à la Goswami-Joardar [53].*

This theorem confirms that our deformation method is a proper generalization of the procedure of Goswami and Joardar and not merely a reformulation.

5.3 Conclusion

In this fifth chapter we proved that our new deformation method is a proper generalization of Goswami and Joardar's work by constructing an example of a deformation coming from a non-dimension-preserving unitary fiber functor. We investigated orthogonal quantum groups and their monoidal equivalences. The action of $SU_q(2)$ on the Podleś sphere and a monoidal equivalence of $SU_q(2)$ are the ingredients for a monoidal deformation which turns out to be non-dimension-preserving and hence not a deformation à la Goswami-Joardar.

Chapter 6

Deformation of the quantum isometry group

The goal of this last chapter is to prove that the quantum isometry group (defined by Bhowmick and Goswami in [15]) of the deformation (in the sense of theorem 3.3.8) of a spectral triple is a suitable deformation of the quantum isometry group of the original spectral triple.

This chapter is structured as follows. We start by recalling some concepts and results of [15] in the first section. In the second section, we develop some tools which we need for the proof of the main theorem of this chapter. We develop a procedure to, given a monoidal equivalence $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$ between two compact quantum groups, construct a monoidal equivalence between certain Woronowicz-C^*-subalgebras of $\mathbb{G}_1$ and $\mathbb{G}_2$ and between certain supergroups of $\mathbb{G}_1$ and $\mathbb{G}_2$. In section 6.3, we prove the main result of this chapter: the quantum isometry group of the deformation of a spectral triple is a suitable deformation of the quantum isometry group of the original spectral triple. Finally in the last section, we apply this to the example we constructed in theorem 5.2.1.

6.1 Quantum isometry groups

Definition 6.1.1 (Definition 2.7 in [15]). *An R-twisted spectral triple (of compact type) is given by a triple $(\mathcal{A}, \mathcal{H}, D)$ and an operator R on $\mathcal{H}$ where*

1. *$(\mathcal{A}, \mathcal{H}, D)$ is a compact spectral triple,*
2. *R is a positive (possibly unbounded) invertible operator such that R commutes with D.*

Remark 6.1.2. *We note that in Definition 2.7 in [15], there is a third condition in the definition of R-twisted spectral triple. However in remark 2.11 of [15], the authors state that this third condition is not necessary. Therefore, we gave the definition above.*

Such an operator R is linked with the preservation of a non-commutative analogue of a volume form.

Definition 6.1.3 ([15]). *Let R be a positive invertible operator and $(\mathcal{A}, \mathcal{H}, D)$ an R-twisted spectral triple. Then a compact quantum group $\mathbb{G}$ acting on $(\mathcal{A}, \mathcal{H}, D)$ by orientation-preserving isometries is said to preserve the R-twisted volume if one has*

$$(\tau_R \otimes \mathrm{id})(\alpha_U(x)) = \tau_R(x) 1_{C(\mathbb{G})}$$

for all $x \in \mathcal{E}_D$, where $\tau_R(x) = \mathrm{Tr}(Rx)$ and where $\mathcal{E}_D$ is the *-subalgebra of $B(\mathcal{H})$ generated by the rank-one operators of the form $\eta\xi^*$, η, ξ eigenvectors of D.

In what follows we will denote by $\mathcal{Q}_R(\mathcal{A}, \mathcal{H}, D)$ (or just $\mathcal{Q}_R$) the category of all compact quantum groups acting by R-twisted volume- and orientation-preserving isometries with as morphisms the morphisms of quantum groups which are compatible with the representations on $\mathcal{H}$.

Moreover, one can prove (as is done in [49]) that for every compact quantum group acting by orientation-preserving isometries, there exists an operator R such that the quantum group is an element of $\mathcal{Q}_R$.

Now Goswami and Bhowmick proved in [15] that there exists a universal object in $\mathcal{Q}_R(\mathcal{A}, \mathcal{H}, D)$.

Theorem 6.1.4 (Theorem 2.14 in [15]). *For any R-twisted spectral triple $(\mathcal{A}, \mathcal{H}, D)$ there exists a universal (initial) object $(\mathrm{QISO}^0_R(\mathcal{A}, \mathcal{H}, D), U_0)$ in the category $\mathcal{Q}_R$. The representation U_0 is faithful.*

We want to note that the preservation of the R-twisted volume is essential. As Bhowmick and Goswami note in the introduction of [15], without this assumption, there does not exist a quantum isometry group for all spectral triples. For example if $\mathcal{A} = M_n(\mathbb{C}), \mathcal{H} = \mathbb{C}^n, D = I$, then the quantum isometry group (if it would exist) would be the equal to the quantum automorphism group (defined in [97]) and in [97], Wang proves that this quantum automorphism group does not exist if one does not impose that a functional should be preserved. This functional will be the imposed volume form.

For notational convenience, we will write $\mathrm{QISO}^0_{\mathrm{R}}$ if there is no confusion possible about the spectral triple. We know moreover that U_0 is faithful from theorem 6.1.4. However, in general α_{U_0} may not be faithful even if U_0 is so. Therefore one has the following definition.

Definition 6.1.5 (Definition 2.16 in [15]). *Let $\mathcal{C} = C^*(\{(f \otimes \mathrm{id})\alpha_{U_0}(a) \mid a \in \mathcal{A}, f \in \mathcal{A}^*\})$ be the C^*-subalgebra of $C(\mathrm{QISO}^0_{\mathrm{R}})$ generated by elements of the form $(f \otimes \mathrm{id})\alpha_{U_0}(a)$, $a \in \mathcal{A}$.*

Then $\mathcal{C}$ is a Woronowicz C^*-subalgebra of $\mathrm{QISO}^0_{\mathrm{R}}$ and the compact quantum group

$$\mathrm{QISO}_R(\mathcal{A}, \mathcal{H}, D) = (\mathcal{C}, \Delta_{\mathrm{QISO}^0_{\mathrm{R}}|_{\mathcal{C}}})$$

is called the quantum group of R-twisted volume- and orientation-preserving isometries or simply the quantum isometry group.

With the deformation method developed in chapter 3, it is a natural question to ask whether the quantum isometry group of a deformed spectral triple is the deformation of the quantum isometry group of the original spectral triple. Or to be more precise: If $(\mathcal{A}, \mathcal{H}, D)$ is an R-twisted spectral triple and $\varphi : \mathrm{QISO}_R(\mathcal{A}, \mathcal{H}, D) \to \mathbb{G}_2$ is a monoidal equivalence, does there exist an operator $\tilde{R}$ such that $(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$ is an $\tilde{R}$-spectral triple and $\mathbb{G}_2 \cong \mathrm{QISO}_{\tilde{R}}(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$. We will prove the positive answer in section 6.3.

But to be able to do so, we need some tools. In the following section, we describe, given a monoidal equivalence $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$, how to construct a monoidal equivalence between certain Woronowicz-C^*-subalgebras (subsection 6.2.1) resp. quantum supergroups (subsection 6.2.3) of $\mathbb{G}_1$ and $\mathbb{G}_2$.

6.2 Tools to induce monoidal equivalences on other quantum groups

6.2.1 Inducing monoidal equivalences on Woronowicz-C^*-subalgebras

Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group. Then we have the following definition:

Definition 6.2.1 ([1]). *Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group and A a C^*-subalgebra of $C(\mathbb{G})$ such that $\Delta(A) \subset A \otimes A$ and $[\Delta_{|_A}(A)(A \otimes 1)] = A \otimes A = [\Delta_{|_A}(A)(1 \otimes A)]$. Then A is called a Woronowicz C^*-subalgebra. We will write $\mathbb{A} = (A, \Delta_{|_A})$ to denote the compact quantum group.*

It is good to remark that the notion of compact quantum quotient group introduced in [95] (see also definition 6.2.11) is a special case of a Woronowicz C^*-subalgebra. However not all Woronowicz C^*-subalgebras are compact quantum quotient groups. We will focus on quantum quotient groups in subsection 6.2.2.

For the rest of this subsection let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a CQG and A a Woronowicz C^*-subalgebra of $\mathbb{G}$. In order to define a unitary fiber functor on $\mathbb{A}$, it is good to examine its representations. We have the following result, which can be easily proved by checking the definitions.

Proposition 6.2.2. *Every representation U of $\mathbb{A} = (A, \Delta_{|_A})$ on a Hilbert space $\mathcal{H}$ is a representation of $\mathbb{G}$ and every representation V of $\mathbb{G}$ is a representation of A if and only if $V \in \mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes A) \subset \mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes C(\mathbb{G}))$.*

To distinguish, we will write $U_{\mathbb{G}}$ for a representation U of $\mathbb{A}$ seen as representation of $\mathbb{G}$. Moreover, we have the following proposition

Proposition 6.2.3. *Let U be a unitary representation of $\mathbb{A}$. Then U is irreducible if and only if $U_{\mathbb{G}}$ is irreducible.*

Proof. Recall the definition $\mathrm{Mor}(V_1, V_2) = \{T \in B(\mathcal{H}_2, \mathcal{H}_1) \mid (T \otimes \mathrm{id})V_2 = V_1(T \otimes \mathrm{id})\}$, where V_1 resp. V_2 are representations of $\mathbb{G}$ on $\mathcal{H}_1$ resp. $\mathcal{H}_2$. We know now that U resp. $U_{\mathbb{G}}$ is irreducible if and only if $\mathrm{Mor}(U, U)$ resp. $\mathrm{Mor}(U_{\mathbb{G}}, U_{\mathbb{G}})$ equals $\mathbb{C}1_{B(\mathcal{H})}$. As it is directly clear that $\mathrm{Mor}(U, U) = \mathrm{Mor}(U_{\mathbb{G}}, U_{\mathbb{G}})$, the proposition is proved. $\square$

Analogously as before, we will write $x_{\mathbb{G}}$ if we look at the equivalence class $x \in \text{Irred}(\mathbb{A})$ seen as equivalence class in $\text{Irred}(\mathbb{G})$. Now let $\mathbb{G}_1, \mathbb{G}_2$ be compact quantum groups and $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$ a monoidal equivalence between them. Suppose moreover that A_1 is a Woronowicz subalgebra of $\mathbb{G}_1$. Then we can construct a unitary fiber functor on $\mathbb{A}_1 = (A_1, \Delta_{|A_1})$ by restricting φ to the representations of $\mathbb{A}_1$ and proof it is a monoidal equivalence between $\mathbb{A}_1$ and a compact quantum group $\mathbb{A}_2$ such that $C(\mathbb{A}_2)$ is a Woronowciz C^*-subalgebra of $\mathbb{G}_2$.

Proposition 6.2.4. *Let $\mathbb{G}_1$ be a compact quantum group, A_1 a Woronowicz C^*-subalgebra of $\mathbb{G}_1$ and ψ a unitary fiber functor on $\mathbb{G}_1$. Then there exists a unitary fiber functor ψ' on $\mathbb{A}_1 = (A_1, \Delta_{1|A_1})$ such that $\psi'(x) = \psi(x_{\mathbb{G}_1})$ for all $x \in \text{Irred}(\mathbb{A}_1)$.*

Proof. Let $x \in \text{Irred}(\mathbb{A}_1)$. Define $\mathcal{H}_{\psi'(x)}$ to be $\mathcal{H}_{\psi(x_{\mathbb{G}})}$ and $\psi'(S) = \psi(S)$ for every $S \in \text{Mor}(y_1 \otimes \ldots \otimes y_k, x_1 \otimes \ldots \otimes x_r), y_1, \ldots, y_k, x_1, \ldots, x_r \in \text{Irred}(\mathbb{A}_1)$. As ψ is a unitary fiber functor, ψ' will satisfy all the necessary conditions to be a unitary fiber functor as well. $\square$

Denoting by $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$ the monoidal equivalence associated to ψ, we can see $C(\mathbb{G}_2)$ as the C^*-algebra generated (as vector space) by the coefficients of the $U^{\varphi(x)}, x \in \text{Irred}(\mathbb{G}_1)$. Now we can define A_2 as the C^*-algebra generated (as vector space) by the coefficients of the $U^{\varphi(x_{\mathbb{G}_1})}, x \in \text{Irred}(\mathbb{A}_1)$. Equivalently,

$$A_2 = [(\omega \otimes \text{id})U^{\varphi(x_{\mathbb{G}_1})} | x \in \text{Irred}(\mathbb{A}_1)]$$

and we also write

$$\mathcal{A}_2 = \langle (\omega \otimes \text{id})U^{\varphi(x_{\mathbb{G}_1})} | x \in \text{Irred}(\mathbb{A}_1) \rangle.$$

Note that they are indeed algebras: if $x, y \in \text{Irred}(\mathbb{A}_1)$, then the product of a matrix coefficient of $U^{\varphi(x)}$ and one of $U^{\varphi(y)}$ is a matrix coefficient of the tensor product of them. By construction this tensor product is a direct sum of representations $\varphi(z_i)$ where all z_i are in $\text{Irred}(\mathbb{A}_1)$.
Now it is clear that ψ' induces a monoidal equivalence φ' between $\mathbb{A}_1$ and a compact quantum group with algebra A_2.

Theorem 6.2.5. *With the map $\Delta_2' = \Delta_{2|A_2}$, $\mathbb{A}_2 = (A_2, \Delta_2')$ is a compact quantum group. Moreover the monoidal equivalence φ', induced by ψ is an equivalence between $\mathbb{A}_1$ and $\mathbb{A}_2$.*

Proof. Written differently, A_2 is the closed linear span of the elements $u_{ij}^{\varphi(x_{\mathbb{G}_1})}, x \in \text{Irred}(\mathbb{A}_1)$. Hence, we get:

$$\Delta_2(u_{ij}^{\varphi(x_{\mathbb{G}_1})}) = \sum_k u_{ik}^{\varphi(x_{\mathbb{G}_1})} \otimes u_{kj}^{\varphi(x_{\mathbb{G}_1})}$$

and as $x \in \mathrm{Irred}(\mathbb{A}_1)$, we see that $\Delta_2(A_2) \subset A_2 \otimes A_2$ and $S(\mathcal{A}_2) \subset \mathcal{A}_2$. Now denote by Δ_2', ε' and S' the restrictions of the coproduct Δ_2, counit ε and antipode S of $\mathbb{G}_2$ defined on $\mathcal{O}(\mathbb{G}_2)$ to $\mathcal{A}_2$. Then $\mathcal{A}_2 = \langle (\omega \otimes \mathrm{id}) U^{\varphi(x_{\mathbb{G}_1})} | x \in \mathrm{Irred}(\mathbb{A}_1) \rangle = \mathcal{O}(\mathbb{A}_2)$ is a Hopf *-algebra which is dense in A_2. This proves that $\mathbb{A}_2 = (A_2, \Delta_2')$ is indeed a compact quantum group. By construction of φ', it is evident that it is a monoidal equivalence between $\mathbb{A}_1$ and $\mathbb{A}_2$. $\square$

Before we go the next subsection, we want to explore how the ($\mathbb{A}_1$-$\mathbb{A}_2$)-bi-Galois object is related to the ($\mathbb{G}_1$-$\mathbb{G}_2$)-bi-Galois object.

Theorem 6.2.6. *Let $\mathbb{G}_1, \mathbb{G}_2, \mathbb{A}$ be compact quantum groups such that $C(\mathbb{A})$ is a Woronowicz C^*-subalgebra of $C(\mathbb{G}_1)$ and such that $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$ is a monoidal equivalence. Let $\mathcal{B}$ be the ($\mathbb{G}_1$-$\mathbb{G}_2$)-bi-Galois object with coaction $\beta_1 : \mathcal{B} \to \mathcal{O}(\mathbb{G}_1) \odot \mathcal{B}$. Let φ' be the monoidal equivalence between $\mathbb{A}_1$ and $\mathbb{A}_2$ as defined in theorem 6.2.5 and define $\mathcal{B}'$ to be the ($\mathbb{A}_1$-$\mathbb{A}_2$)-bi-Galois object with coaction $\gamma_1 : \mathcal{B}' \to \mathcal{O}(\mathbb{A}_1) \odot \mathcal{B}'$. Then we have*

$$\mathcal{B}' = \{ b \in \mathcal{B} | \beta_1(b) \in \mathcal{O}(\mathbb{A}_1) \odot \mathcal{B} \}$$

and $\gamma_1 = \beta_{1|_{\mathcal{B}'}}$.

Proof. From the original proof of theorem 2.6.5 (which is theorem 3.9 in [27]), we know that $\mathcal{B}' = \oplus_{x \in \mathrm{Irred}(\mathbb{A}_1)} B(\mathcal{H}_{\varphi(x)}, \mathcal{H}_x)^*$ and $\mathcal{B} = \oplus_{x \in \mathrm{Irred}(\mathbb{G}_1)} B(\mathcal{H}_{\varphi(x)}, \mathcal{H}_x)^*$. Hence $\mathcal{B}' \hookrightarrow \mathcal{B}$. Also, $X^x \in B(\mathcal{H}_{\varphi(x)}, \mathcal{H}_x) \odot \mathcal{B}$ is defined such that $(\omega_x \otimes \mathrm{id})(X^x) = (\delta_{x,y}\omega_x)_{y \in \mathrm{Irred}(\mathbb{G}_1)}$ for all $\omega_x \in B(\mathcal{H}_{\varphi(x)}, \mathcal{H}_x)^*$. By definition, we see that for $x \in \mathrm{Irred}(\mathbb{A}_1)$, $X^x = X^{x_{\mathbb{G}_1}}$. As β_1 resp. γ_1 are defined by $(\mathrm{id} \otimes \beta_1)(X^x) = U_{12}^x X_{13}^x$ ($x \in \mathrm{Irred}(\mathbb{G}_1)$) resp. $(\mathrm{id} \otimes \gamma_1)(X^x) = U_{12}^x X_{13}^x$ ($x \in \mathrm{Irred}(\mathbb{A}_1)$), it follows directly that $\gamma_1 = (\beta_1)_{|_{\mathcal{B}'}}$. Moreover, if $x \in \mathrm{Irred}(\mathbb{A}_1)$, $U_{12}^x X_{13}^x \in B(\mathcal{H}_{\varphi(x)}, \mathcal{H}_x) \odot \mathcal{O}(\mathbb{A}_1) \odot \mathcal{B}$ and hence $\beta_1(b) \in \mathcal{O}(\mathbb{A}_1) \odot \mathcal{B}$ for $b \in \mathcal{B}'$. If $x \in \mathrm{Irred}(\mathbb{G}_1)$ but $x \notin \mathrm{Irred}(\mathbb{A}_1)$, $U_{12}^x X_{13}^x \notin B(\mathcal{H}_{\varphi(x)}, \mathcal{H}_x) \odot \mathcal{O}(\mathbb{A}_1) \odot \mathcal{B}$ and hence for $b \in \mathcal{B}$ but $b \notin \mathcal{B}', \beta_1(b) \notin \mathcal{O}(\mathbb{A}_1) \odot \mathcal{B}$. This concludes the proof. $\square$

Remark 6.2.7. *In the special case of compact quantum quotient groups, a compact quantum quotient group of $\mathbb{G}_1$ will be monoidally equivalent with a compact quantum group which has as algebra a Woronowicz C^*-subalgebra of $\mathbb{G}_2$. Whether that compact quantum group is a compact quantum quotient group as well is still unknown [95]. This is the subject of the next subsection.*

6.2.2 Normal quantum subgroups and quantum quotients

We first recall the definition of a quantum subgroup

Definition 6.2.8 ([79, 95], definition 2.2.15)**.** *Let $\mathbb{G} = (C_u(\mathbb{G}), \Delta_{\mathbb{G}})$ and $\mathbb{H} = (C_u(\mathbb{H}), \Delta_{\mathbb{H}})$ be compact quantum groups equipped with their universal C^*-norms. Suppose moreover that there exists a surjective map $\theta : C_u(\mathbb{G}) \to C_u(\mathbb{H})$ satisfying $\Delta_{\mathbb{H}} \circ \theta = (\theta \otimes \theta)\Delta_{\mathbb{G}}$. Then we call $\mathbb{H}$ a quantum subgroup of $\mathbb{G}$. Equivalently, $\mathbb{G}$ is called a quantum supergroup of $\mathbb{H}$.*

Note that in the case $C_u(\mathbb{G})$ is commutative, a quantum subgroup of $\mathbb{G}$ is a subgroup in the classical sense by Gelfand Naimark theory. Moreover, given a compact group and a subgroup, we can make the quotient space. In the quantum setting the quotient space is defined in [98].

Definition 6.2.9 ([95])**.** *Let $\mathbb{G}$ be a compact quantum group and $\mathbb{H}$ a quantum subgroup of $\mathbb{G}$. Then*

$$C(\mathbb{G}/\mathbb{H}) = \{x \in C(\mathbb{G}) \mid (\mathrm{id} \otimes \theta)\Delta(x) = x \otimes 1_{C_u(\mathbb{H})}\}$$

$$C(\mathbb{H}\backslash\mathbb{G}) = \{x \in C(\mathbb{G}) \mid (\theta \otimes \mathrm{id})\Delta(x) = 1_{C_u(\mathbb{H})} \otimes x\}$$

are the right resp. left quotient space. Similarly $\mathcal{O}(\mathbb{H})\backslash\mathbb{G}$ and $\mathcal{O}(\mathbb{G}/\mathbb{H})$ are defined to be $\mathcal{O}(\mathbb{H})\backslash\mathbb{G} := C(\mathbb{H}\backslash\mathbb{G}) \cap \mathcal{O}(\mathbb{G})$ and $\mathcal{O}(\mathbb{G}/\mathbb{H}) = C(\mathbb{G}/\mathbb{H}) \cap \mathcal{O}(\mathbb{G})$.

Note that if $\mathbb{G}$ and $\mathbb{H}$ are classical groups, the above notions are the classical definitions for quotients of groups by subgroups. Inspired by this classical case, Wang defined what it means for $\mathbb{H}$ to be normal.

Proposition 6.2.10 ([95])**.** *Let $\mathbb{G}$ be a compact quantum group and $\mathbb{H}$ a quantum subgroup. Then the following conditions are equivalent*

- *$C(\mathbb{G}/\mathbb{H})$ is a C^*-subalgebra of $C(\mathbb{G})$ and $(C(\mathbb{G}/\mathbb{H}), \Delta_{|_{C(\mathbb{G}/\mathbb{H})}})$ is a compact quantum group*
- *$C(\mathbb{H}\backslash\mathbb{G})$ is a C^*-subalgebra of $C(\mathbb{G})$ and $(C(\mathbb{H}\backslash\mathbb{G}), \Delta_{|_{C(\mathbb{H}\backslash\mathbb{G})}})$ is a compact quantum group*
- *$C(\mathbb{G}/\mathbb{H}) = C(\mathbb{H}\backslash\mathbb{G})$*

Definition 6.2.11 ([95])**.** *Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group and $\mathbb{H}$ a quantum subgroup of $C(\mathbb{G})$. Then if one of the equivalent conditions in proposition 6.2.10 is satisfied, $\mathbb{H}$ is called a normal quantum subgroup of $\mathbb{G}$ and $\mathbb{G}/\mathbb{H}$ a quantum quotient group.*

By definition, if $\mathbb{H}$ is a normal quantum subgroup of a compact quantum group $\mathbb{G}$, then $C(\mathbb{G}/\mathbb{H})$ is a Woronowicz C^*-subalgebra. Therefore, we can apply the theorems of the previous subsection and investigate what it means in this context.

Representations of quotients of compact quantum groups

Let $\mathbb{G}$ be a compact quantum group, $\mathbb{H}$ a normal quantum subgroup with $\theta : C(\mathbb{G}) \to C(\mathbb{H})$. We will investigate what the (irreducible) representations of $\mathbb{G}/\mathbb{H}$ are, applying proposition 6.2.2.

Proposition 6.2.12. *Let U be a unitary representation of $\mathbb{G}$ on a Hilbert space $\mathcal{H}$. Then U is a unitary representation of $\mathbb{G}/\mathbb{H}$ if and only if $(\mathrm{id} \otimes \theta)U = \mathrm{id} \otimes 1_{C(\mathbb{H})}$. Moreover, every unitary representation of $\mathbb{G}/\mathbb{H}$ is of this form.*

Proof. If U is a unitary representation of $\mathbb{G}$, by proposition 6.2.2, it suffices to prove that $U \in \mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes C(\mathbb{G}/\mathbb{H}))$ if and only if $(\mathrm{id}_{\mathcal{H}} \otimes \theta)U = \mathrm{id}_{\mathcal{H}} \otimes 1_{C(\mathbb{H})}$. Note now that

$$(\mathrm{id}_{\mathcal{H}} \otimes \theta \otimes \mathrm{id}_{C(\mathbb{G})})(\mathrm{id}_{\mathcal{H}} \otimes \Delta)U = (\mathrm{id}_{\mathcal{H}} \otimes \theta \otimes \mathrm{id}_{C(\mathbb{G})})U_{12}U_{13} = ((\mathrm{id}_{\mathcal{H}} \otimes \theta)U)_{12}U_{13}.$$

Hence $U \in \mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes C(\mathbb{G}/\mathbb{H}))$ if and only if $((\mathrm{id}_{\mathcal{H}} \otimes \theta)U)_{12}U_{13} = U_{13}$ or equivalently, $(\mathrm{id}_{\mathcal{H}} \otimes \theta)U = \mathrm{id}_{\mathcal{H}} \otimes 1_{C_u(\mathbb{H})}$ which concludes the proof. $\square$

Inducing monoidal equivalences on quantum quotient groups

Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two compact quantum groups and $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$ a monoidal equivalence between them. Suppose moreover that $\mathbb{H}$ is a normal quantum subgroup of $\mathbb{G}_1$. Applying the procedure of the previous section, one sees that again, we can construct a compact quantum group $\mathbb{G}_2'$ which is monoidally equivalent with $\mathbb{G}_1/\mathbb{H}$ and such that $C(\mathbb{G}_2')$ is a Woronowicz-C^*-subalgebra of $\mathbb{G}_2$.

Whether $\mathbb{G}_2'$ is a quantum quotient group of $\mathbb{G}_2$ remains an open problem. We see that $C(\mathbb{G}_2')$ is indeed a Woronowicz subalgebra of $C(\mathbb{G}_2)$ but, unlike in the classical case, that is not sufficient for $\mathbb{G}_2'$ to be a quantum quotient. If $\mathbb{G}_2$ has property F (i.e. a quantum group $\mathbb{G}$ has property F if every Woronowicz subalgebra of $\mathbb{G}$ is a quantum quotient group) introduced by Wang in [98], it is indeed a quantum quotient group.

Finally, we have a look at the ($\mathbb{G}_1/\mathbb{H}$-$\mathbb{G}_2'$)-bi-Galois object.

Theorem 6.2.13. *Let $\mathbb{G}_1, \mathbb{G}_2, \mathbb{H}$ be compact quantum groups such that $\mathbb{H}$ is a normal quantum subgroup of $\mathbb{G}_1$ with $\theta : C_u(\mathbb{G}_1) \to C_u(\mathbb{H})$ the surjective morphism. Suppose moreover that $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$ is a monoidal equivalence. Let $\mathcal{B}$ be the ($\mathbb{G}_1$-$\mathbb{G}_2$)-bi-Galois object with coaction $\beta_1 : \mathcal{B} \to \mathcal{O}(\mathbb{G}_1) \odot \mathcal{B}$. Let*

φ' be the monoidal equivalence between $\mathbb{G}_1/\mathbb{H}$ and $\mathbb{G}_2'$ and define $\mathcal{B}'$ to be the ($\mathbb{G}_1/\mathbb{H}$-$\mathbb{G}_2'$)-bi-Galois object. Then we have

$$\mathcal{B}' = \{b \in \mathcal{B} | (\theta \otimes \mathrm{id})\beta_1(b) = 1_{C_u(\mathbb{H})} \otimes b\}.$$

Proof. By proposition 6.2.6, it suffices to prove that

$$\{b \in \mathcal{B} | (\theta \otimes \mathrm{id})\beta_1(b) = 1_{C_u(\mathbb{H})} \otimes b\} = \{b \in \mathcal{B} | \beta_1(b) \in \mathcal{O}(\mathbb{G}_1/\mathbb{H}) \odot \mathcal{B}\}.$$

Now let $b \in \mathcal{B}$ such that $(\theta \otimes \mathrm{id})\beta_1(b) = 1_{C_u(\mathbb{H})} \otimes b$. Then

$$\begin{aligned}(\theta \otimes \mathrm{id}_{C(\mathbb{G}_1)} \otimes \mathrm{id}_{\mathcal{B}})(\Delta \otimes \mathrm{id}_{\mathcal{B}})\beta_1(b) &= (\theta \otimes \mathrm{id}_{C(\mathbb{G})} \otimes \mathrm{id}_{\mathcal{B}})(\mathrm{id}_{C(\mathbb{G}_1)} \otimes \beta_1)\beta_1(b) \\ &= 1_{C_u(\mathbb{H})} \otimes \beta_1(b)\end{aligned}$$

which proves $\beta_1(b) \in \mathcal{O}(\mathbb{G}_1/\mathbb{H}) \odot \mathcal{B}$. Conversely, suppose that $\beta_1(b) \in \mathcal{O}(\mathbb{G}_1/\mathbb{H}) \odot \mathcal{B}$, then

$$(\theta \otimes \mathrm{id}_{C(\mathbb{G}_1)} \otimes \mathrm{id}_{\mathcal{B}})(\Delta \otimes \mathrm{id}_{\mathcal{B}})\beta_1(b) = 1 \otimes \beta_1(b)$$

and hence

$$1 \otimes b = (\theta \otimes \varepsilon \otimes \mathrm{id}_{\mathcal{B}})(\Delta \otimes \mathrm{id}_{\mathcal{B}})\beta_1(b) = (\theta \otimes \mathrm{id})\beta_1(b)$$

concluding proof. □

6.2.3 Inducing monoidal equivalences on supergroups

In this subsection we describe, given a monoidal equivalence $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$, how to construct a monoidal equivalence between certain quantum supergroups of $\mathbb{G}_1$ and $\mathbb{G}_2$.
So, let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two compact quantum groups and let $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$ be a monoidal equivalence. Moreover suppose $\mathbb{G}_1$ is a compact quantum subgroup of a compact quantum group $\mathbb{H}_1$. As we have done in subsection 6.2.1 for Woronowicz C^*-subalgebras, we will describe a method to construct a unitary fiber functor on $\mathbb{H}_1$ from the monoidal equivalence φ.

Let $\pi : C_u(\mathbb{H}_1) \to C_u(\mathbb{G}_1)$ be the surjective morphism which is compatible with the quantum group structure. Now note that for a representation U of $\mathbb{H}_1$ on a Hilbert space $\mathcal{H}$, $(\mathrm{id}_{\mathcal{H}} \otimes \pi)U$ is a representation of $\mathbb{G}_1$. Therefore, for $x \in \mathrm{Irred}(\mathbb{H}_1)$ define $x_{\mathbb{G}_1}$ to be the equivalence class of $(\mathrm{id}_{\mathcal{H}} \otimes \pi)U^x$ as representation of $\mathbb{G}_1$ and

- if $(\mathrm{id} \otimes \pi)U^x$ is irreducible, let $\mathcal{H}_{x_{\mathbb{G}_1}} = \mathcal{H}_x$;

- If $(\mathrm{id}\otimes\pi)U^x$ is reducible, say $(\mathrm{id}\otimes\pi)U^x = \oplus_{i=1}^n U^{y_i}$, $y_i \in \mathrm{Irred}(\mathbb{G}_1)$, then let $\mathcal{H}_{x_{\mathbb{G}_1}} = \oplus_{i=1}^n \mathcal{H}_{y_i}$.

If $x^1, \ldots, x^r, y^1, \ldots, y^s$ are classes of irreducible representations of $\mathbb{H}_1$ with $U^{x^i_{\mathbb{G}_1}} = \oplus_{j_i} U^{z^i_{j_i}}$ and $U^{y^i_{\mathbb{G}_1}} = \oplus_{k_i} U^{t^i_{k_i}}$, we denote for a morphism $S \in \mathrm{Mor}(x_1 \otimes \ldots \otimes x_r, y_1 \otimes \ldots, y_s)$, $S_{\mathbb{G}_1} = \bigoplus_{j_1,\ldots,j_r,k_1,\ldots,k_s} S^{j_1,\ldots,j_r}_{k_1,\ldots,k_s}$ to be the morphism S but seen as element of $\bigoplus_{j_1,\ldots,j_r,k_1,\ldots,k_s} \mathrm{Mor}(z^1_{k_1} \otimes \ldots \otimes z^s_{k_s}, t^1_{j_1} \otimes \ldots \otimes t^r_{j_r})$, i.e. $S^{j_1,\ldots,j_r}_{k_1,\ldots,k_s} \in \mathrm{Mor}(z^1_{k_1} \otimes \ldots \otimes z^s_{k_s}, t^1_{j_1} \otimes \ldots \otimes t^r_{j_r})$.

Then we can define the following map:

Proposition 6.2.14. *Let $\mathbb{G}_1, \mathbb{G}_2, \mathbb{H}_1$ and φ be as above. For $x \in \mathrm{Irred}(\mathbb{H}_1)$ with $U^{x_{\mathbb{G}_1}} = (\mathrm{id}\otimes\pi)U^x = \oplus_{i=1}^n U^{y_i}$, $y_i \in \mathrm{Irred}(\mathbb{G}_1)$ define $\mathcal{H}_{\psi'(x)} = \oplus_{i=1}^n \mathcal{H}_{\varphi(y_i)}$ and for $S \in \mathrm{Mor}(x_1 \otimes \ldots \otimes x_r, y_1 \otimes \ldots, y_s)$ with $S_{\mathbb{G}_1} = \bigoplus_{j_1,\ldots,j_r,k_1,\ldots,k_s} S^{j_1,\ldots,j_r}_{k_1,\ldots,k_s}$, let $\psi'(S) = \bigoplus_{j_1,\ldots,j_r,k_1,\ldots,k_s} \varphi(S^{j_1,\ldots,j_r}_{k_1,\ldots,k_s})$. Then the collection of maps*

$$\mathcal{H}_x \mapsto \mathcal{H}_{\psi'(x)} \qquad S \in \mathrm{Mor}(x_1 \otimes \ldots \otimes x_r, y_1 \otimes \ldots, y_s) \mapsto \psi'(S)$$

constitutes a unitary fiber functor ψ' on $\mathbb{H}_1$.

The proof follows directly by construction of $\mathcal{H}_{\psi'}$ and $\psi'(S)$. By theorem 2.6.4, there exists a compact quantum group $\mathbb{H}_2$ and a monoidal equivalence $\varphi' : \mathbb{H}_1 \to \mathbb{H}_2$. In theorem 6.2.15 we will describe the bi-Galois object associated to φ and the compact quantum group $\mathbb{H}_2$ explicitly.

Theorem 6.2.15. *Let $\mathbb{G}_1, \mathbb{G}_2, \mathbb{H}_1$ be compact quantum groups such that $\mathbb{G}_1$ is a compact quantum subgroup of $\mathbb{H}_1$ with surjective morphism $\pi : C_u(\mathbb{H}_1) \to C_u(\mathbb{G}_1)$. Let $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$ be a monoidal equivalence as above and let $\mathbb{H}_2$ and $\varphi' : \mathbb{H}_1 \to \mathbb{H}_2$ be the compact quantum group and monoidal equivalence induced by φ by propositions 6.2.14 and 2.6.4. Denoting by $\mathcal{B}$ the ($\mathbb{G}_1$-$\mathbb{G}_2$)-bi-Galois object associated to φ, by $\tilde{\mathcal{B}}$ the ($\mathbb{G}_2$-$\mathbb{G}_1$)-bi-Galois object associated to φ^{-1} and by $\mathcal{B}'$ the ($\mathbb{H}_1$-$\mathbb{H}_2$)-bi-Galois object associated to φ', we have*

$$\mathcal{B}' \cong \mathcal{O}(\mathbb{H}_1) \underset{\mathcal{O}(\mathbb{G}_1)}{\boxdot} \mathcal{B} \tag{6.2.1}$$

and

$$\mathcal{O}(\mathbb{H}_2) \cong \tilde{\mathcal{B}} \underset{\mathcal{O}(\mathbb{G}_1)}{\boxdot} \mathcal{O}(\mathbb{H}_1) \underset{\mathcal{O}(\mathbb{G}_1)}{\boxdot} \mathcal{B} \tag{6.2.2}$$

using the right resp. left coactions $(\mathrm{id}\otimes\pi)\Delta_{\mathbb{H}_1} : \mathcal{O}(\mathbb{H}_1) \to \mathcal{O}(\mathbb{H}_1) \odot \mathcal{O}(\mathbb{G}_1)$ resp. $(\pi \otimes \mathrm{id})\Delta_{\mathbb{H}_1} : \mathcal{O}(\mathbb{H}_1) \to \mathcal{O}(\mathbb{G}_1) \odot \mathcal{O}(\mathbb{H}_1)$ of $\mathcal{O}(\mathbb{G}_1)$ on $\mathcal{O}(\mathbb{H}_1)$.

Proof. Let X^x, $x \in \mathrm{Irred}(\mathbb{G}_1)$ be the unitaries from theorem 2.6.5 associated to φ. Define for $x \in \mathrm{Irred}(\mathbb{H}_1)$, $X^{x_{\mathbb{G}_1}} = \oplus_{i=1}^n X^{y_i}$ if $U^{x_{\mathbb{G}_1}} = (\mathrm{id} \otimes \pi)U^x = \oplus_{i=1}^n U^{y_i}$, $y_i \in \mathrm{Irred}(\mathbb{G}_1)$. Moreover define for $x \in \mathrm{Irred}(\mathbb{H}_1)$,

$$Y^x = U^x_{12} X^{x_{\mathbb{G}_1}}_{13} \in B(\mathcal{H}_{\varphi'(x)}, \mathcal{H}_x) \odot \mathcal{O}(\mathbb{H}_1) \odot \mathcal{B}. \tag{6.2.3}$$

We claim that the Y^x with the functional $\omega' = h_{\mathbb{H}_1} \otimes \omega$ ($h_{\mathbb{H}_1}$ is the Haar state of $\mathbb{H}_1$) satisfy the properties 1(a), 1(b) and 1(c) of theorem 2.6.5 applied to φ'. Indeed, we have for $x, y, z \in \mathrm{Irred}(\mathbb{H}_1)$ and $S \in \mathrm{Mor}(y \otimes z, x)$

$$\begin{aligned}
Y^y_{13} Y^z_{23} (\varphi'(S) \otimes \mathrm{id}) &= U^y_{13} X^{y_{\mathbb{G}_1}}_{14} U^z_{23} X^{z_{\mathbb{G}_1}}_{24} (\varphi'(S) \otimes \mathrm{id}) \\
&= U^y_{13} U^z_{23} X^{y_{\mathbb{G}_1}}_{14} X^{z_{\mathbb{G}_1}}_{24} (\varphi'(S) \otimes \mathrm{id}) \\
&= U^y_{13} U^z_{23} (S \otimes \mathrm{id}) X^{x_{\mathbb{G}_1}}_{13} \\
&= (S \otimes \mathrm{id}) U^x_{12} X^{x_{\mathbb{G}_1}}_{13} \\
&= (S \otimes \mathrm{id}) Y^x.
\end{aligned}$$

Moreover $(\mathrm{id} \otimes \omega') Y^x = (\mathrm{id} \otimes h_{\mathbb{H}_1} \otimes \omega)(U^x_{12} X^{x_{\mathbb{G}_1}}_{13}) = 0$ if $x \neq \varepsilon$.

Hence to prove (6.2.1) it suffices to prove that the matrix coefficients of the Y^x constitute a linear basis of $\mathcal{O}(\mathbb{H}_1) \underset{\mathcal{O}(\mathbb{G}_1)}{\boxdot} \mathcal{B}$. Note first that the matrix coefficients of the Y^x are elements of $\mathcal{O}(\mathbb{H}_1) \underset{\mathcal{O}(\mathbb{G}_1)}{\boxdot} \mathcal{B}$. Indeed,

$$\big(\mathrm{id} \otimes (\mathrm{id}_{\mathcal{O}(\mathbb{H}_1)} \otimes \pi) \Delta_{\mathbb{H}_1} \otimes \mathrm{id}_{\mathcal{B}}\big) U^x_{12} X^{x_{\mathbb{G}_1}}_{13} = U^x_{12} U^{x_{\mathbb{G}_1}}_{13} X^{x_{\mathbb{G}_1}}_{14} = (\mathrm{id} \otimes \mathrm{id}_{\mathcal{O}(\mathbb{H}_1)} \otimes \beta_1) U^x_{12} X^{x_{\mathbb{G}_1}}_{13}.$$

Moreover, as every irreducible representation of $\mathbb{G}_1$ is a subrepresentation of some $x_{\mathbb{G}_1}$, $x \in \mathrm{Irred}(\mathbb{H}_1)$, the matrix coefficients of the $X^{x_{\mathbb{G}_1}}$ resp. the U^x form a basis of $\mathcal{B}$ resp. $\mathcal{O}(\mathbb{H}_1)$. Hence, the matrix coefficients of the Y^x are linearly independent. Finally we prove that they are also generating. Let z be an arbitrary element of $\mathcal{O}(\mathbb{H}_1) \underset{\mathcal{O}(\mathbb{G}_1)}{\boxdot} \mathcal{B}$. Then z is of the form $\sum \lambda^{ij}_{st} u^x_{ij} \otimes b^y_{st}$ where the u^x_{ij} resp. b^y_{st} are the matrix coefficients of the U^x resp. X^y, $x \in \mathrm{Irred}(\mathbb{H}_1), y \in \mathrm{Irred}(\mathbb{G}_1)$ and $\lambda^{ij}_{st} \in \mathbb{C}$. As $z \in \mathcal{O}(\mathbb{H}_1) \underset{\mathcal{O}(\mathbb{G}_1)}{\boxdot} \mathcal{B}$, $\sum \lambda^{ij}_{st} u^x_{ik} \otimes \pi(u^x_{kj}) \otimes b^y_{st} = \sum \lambda^{ij}_{st} u^x_{ij} \otimes u^y_{sr} \otimes b^y_{rt}$ and hence z is a linear combination of matrix coefficients of $U^x_{12} X^{x_{\mathbb{G}_1}}_{13}$. As the unitaries satisfying properties 1(a), 1(b) and 1(c) of theorem 2.6.5 are unique, the Y^x are those unitaries and $\mathcal{B}' \cong \mathcal{O}(\mathbb{H}_1) \underset{\mathcal{O}(\mathbb{G}_1)}{\boxdot} \mathcal{B}$. This concludes the proof of the first result (6.2.1).

For the second result 6.2.2, let Z^y, $y \in \mathrm{Irred}(\mathbb{G}_2)$ be the unitaries from theorem 2.6.5 associated to φ^{-1}. If $U^{x_{\mathbb{G}_1}} = (\mathrm{id} \otimes \pi)U^x = \oplus_i U^{y_i}$ for $x \in \mathrm{Irred}(\mathbb{H}_1)$, $y_i \in \mathrm{Irred}(\mathbb{G}_1)$, we will denote $U^{\varphi(x_{\mathbb{G}_1})} = \oplus_i U^{\varphi(y_i)}$ and $Z^{\varphi(x_{\mathbb{G}_1})} = \oplus_i Z^{\varphi(y_i)} \in B(\mathcal{H}_x, \mathcal{H}_{\varphi'(x)}) \odot \tilde{\mathcal{B}}$.

Therefore, we can define

$$V^{\varphi'(x)} = Z_{12}^{\varphi(x_{\mathbb{G}_1})} U_{13}^x X_{14}^{x_{\mathbb{G}_1}}.$$

Then, one can prove analogously as above that for $x, y, z \in \mathrm{Irred}(\mathbb{H}_1)$ and $S \in \mathrm{Mor}(y \otimes z, x)$

$$\begin{aligned}
V_{13}^{\varphi'(y)} V_{23}^{\varphi'(z)} (\varphi'(S) \otimes \mathrm{id}) &= Z_{13}^{\varphi(y_{\mathbb{G}_1})} U_{14}^y X_{15}^{y_{\mathbb{G}_1}} Z_{23}^{\varphi(z_{\mathbb{G}_1})} U_{24}^z X_{25}^{z_{\mathbb{G}_1}} (\varphi'(S) \otimes \mathrm{id}) \\
&= Z_{13}^{\varphi(y_{\mathbb{G}_1})} Z_{23}^{\varphi(z_{\mathbb{G}_1})} U_{14}^y U_{24}^z X_{15}^{y_{\mathbb{G}_1}} X_{25}^{z_{\mathbb{G}_1}} (\varphi'(S) \otimes \mathrm{id}) \\
&= Z_{13}^{\varphi(y_{\mathbb{G}_1})} Z_{23}^{\varphi(z_{\mathbb{G}_1})} U_{14}^y U_{24}^z (S \otimes \mathrm{id}) X_{14}^{x_{\mathbb{G}_1}} \\
&= Z_{13}^{\varphi(y_{\mathbb{G}_1})} Z_{23}^{\varphi(z_{\mathbb{G}_1})} (S \otimes \mathrm{id}) U_{13}^x X_{14}^{x_{\mathbb{G}_1}} \\
&= (\varphi'(S) \otimes \mathrm{id}) Z_{12}^{\varphi(x_{\mathbb{G}_1})} U_{13}^x X_{14}^{x_{\mathbb{G}_1}} \\
&= (\varphi'(S) \otimes \mathrm{id}) V^{\varphi'(x)}.
\end{aligned}$$

The argument to prove that the matrix coefficients of $V^{\varphi'(x)}$ form a linear basis of $C(\mathbb{H}_2)$ is the same as in the first part of the proof.

□

Moreover, the newly constructed compact quantum group $\mathbb{H}_2$ is a supergroup of $\mathbb{G}_2$.

Proposition 6.2.16. *We have a surjective morphism of compact quantum groups $\pi' : C_u(\mathbb{H}_2) \to C_u(\mathbb{G}_2)$ such that*

$$(\mathrm{id}_{\mathcal{H}_{\varphi'(x)}} \otimes \pi') V^{\varphi'(x)} = U^{\varphi(x_{\mathbb{G}_1})} \tag{6.2.4}$$

for every $x \in \mathrm{Irred}(\mathbb{H}_1)$ implying that $\mathbb{G}_2$ is a quantum subgroup of $\mathbb{H}_2$.

Proof. The map π' is well defined by (6.2.4) as the matrix coefficients of the $V^{\varphi'(x)}$ constitute a linear basis of $\mathcal{O}(\mathbb{H}_2)$. Moreover, it is a linear surjection and it follows directly that it is coalgebra map. It suffices to prove that π' is an algebra map. Therefore, denoting by $f : \mathcal{O}(\mathbb{G}_2) \to \tilde{B} \underset{\mathcal{O}(\mathbb{G}_1)}{\boxdot} \mathcal{B}$ the isomorphism of proposition 2.6.11 (applied to $\varphi^{-1} : \mathbb{G}_2 \to \mathbb{G}_1$) such that $(\mathrm{id} \otimes f)U^{\varphi(x)} = Z_{12}^{\varphi(x)} X_{13}^{x}$, $x \in \mathrm{Irred}(\mathbb{G}_1)$ it is easy to see that

$$\begin{aligned}
&(\mathrm{id}_{\mathcal{H}_{\varphi'(x)}} \otimes f^{-1})(\mathrm{id}_{\mathcal{H}_{\varphi'(x)}} \otimes \mathrm{id}_{\tilde{\mathcal{B}}} \otimes \varepsilon \otimes \mathrm{id}_{\mathcal{B}})V^{\varphi'(x)} \\
&= (\mathrm{id}_{\mathcal{H}_{\varphi'(x)}} \otimes f^{-1})(\mathrm{id} \otimes \mathrm{id}_{\tilde{\mathcal{B}}} \otimes \varepsilon \otimes \mathrm{id}_{\mathcal{B}})(Z_{12}^{\varphi(x_{\mathbb{G}_1})} U_{13}^{x} X_{14}^{x_{\mathbb{G}_1}}) \\
&= (\mathrm{id}_{\mathcal{H}_{\varphi'(x)}} \otimes f^{-1})(Z_{12}^{\varphi(x_{\mathbb{G}_1})} X_{13}^{x_{\mathbb{G}_1}}) \\
&= \oplus_i (\mathrm{id}_{\mathcal{H}_{\varphi(y_i)}} \otimes f^{-1})(Z_{12}^{\varphi(y_i)} X_{13}^{y_i}) \\
&= \oplus_i U^{\varphi(y_i)} = U^{\varphi(x_{\mathbb{G}_1})} \\
&= (\mathrm{id}_{\mathcal{H}_{\varphi'(x)}} \otimes \pi')V^{\varphi'(x)}
\end{aligned}$$

if $U^{x_{\mathbb{G}_1}} = \oplus_i U^{y_i}$. Hence $(\mathrm{id}_{\mathcal{H}_{\varphi'(x)}} \otimes f^{-1})(\mathrm{id}_{\mathcal{H}_{\varphi'(x)}} \otimes \mathrm{id}_{\tilde{\mathcal{B}}} \otimes \varepsilon \otimes \mathrm{id}_{\mathcal{B}}) = (\mathrm{id}_{\mathcal{H}_{\varphi'(x)}} \otimes \pi')$ proving that π is multiplicative as composition of algebra maps. This concludes the proof. $\square$

Finally we prove that the two monoidal equivalences φ and φ' make isomorphic deformed spectral triples.

Proposition 6.2.17. *Let $\mathbb{G}_1, \mathbb{G}_2, \mathbb{H}_1$ be compact quantum groups such that $\mathbb{G}_1$ is a compact quantum subgroup of $\mathbb{H}_1$ with surjective morphism $\pi : C_u(\mathbb{H}_1) \to C_u(\mathbb{G}_1)$ and let $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$ be a monoidal equivalence as above. Let $\mathbb{H}_2$ and φ' be the compact quantum group and monoidal equivalence induced by φ as in proposition 6.2.14. Suppose $\mathbb{H}_1$ resp. $\mathbb{G}_1$ acts algebraically and by orientation preserving isometries with a unitary representation V resp. U on a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ such that $U = (\mathrm{id} \otimes \pi)V$. Denoting by $\mathcal{B}$ the ($\mathbb{G}_1$-$\mathbb{G}_2$)-bi-Galois object associated to φ, by $\tilde{\mathcal{B}}$ the ($\mathbb{G}_2$-$\mathbb{G}_1$)-bi-Galois object associated to φ^{-1} and by $\mathcal{B}'$ the ($\mathbb{H}_1$-$\mathbb{H}_2$)-bi-Galois object associated to φ', the deformed spectral triples*

$$(\mathcal{A} \underset{\mathcal{O}(\mathbb{G}_1)}{\boxdot} \mathcal{B}, \mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}), \tilde{D})$$

and

$$(\mathcal{A} \underset{\mathcal{O}(\mathbb{H}_1)}{\boxdot} \mathcal{B}', \mathcal{H} \underset{C(\mathbb{H}_1)}{\boxtimes} L^2(\mathcal{B}'), \tilde{D}')$$

(where $\tilde{D}'$ is the deformation of D along φ') are isomorphic.

Proof. It is easy to see that the map

$$\lambda : \mathcal{A} \underset{\mathcal{O}(\mathbb{G}_1)}{\boxdot} \mathcal{B} \to \mathcal{A} \underset{\mathcal{O}(\mathbb{H}_1)}{\boxdot} \mathcal{O}(\mathbb{H}_1) \underset{\mathcal{O}(\mathbb{G}_1)}{\boxdot} \mathcal{B} : z \mapsto (\alpha_V \otimes \mathrm{id}_{\mathcal{B}})z$$

is an isomorphism of *-algebras with inverse $(\mathrm{id}_{\mathcal{A}} \otimes \varepsilon_{\mathbb{H}_1} \otimes \mathrm{id}_{\mathcal{B}})$. Moreover, with

$$T : L^2(\mathcal{O}(\mathbb{H}_1) \underset{\mathcal{O}(\mathbb{G}_1)}{\boxdot} \mathcal{B}) \to L^2(\mathcal{O}(\mathbb{H}_1)) \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})$$

the isomorphism of proposition 3.2.4, let

$$\phi : \mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}) \to \mathcal{H} \underset{C(\mathbb{H}_1)}{\boxtimes} L^2(\mathcal{O}(\mathbb{H}_1) \underset{\mathcal{O}(\mathbb{G}_1)}{\boxdot} \mathcal{B}) : \eta \to (\mathrm{id} \otimes T^{-1})V_{12}\eta_{13}.$$

Then defining

$$\phi' : \mathcal{H} \underset{C(\mathbb{H}_1)}{\boxtimes} L^2(\mathcal{O}(\mathbb{H}_1) \underset{\mathcal{O}(\mathbb{G}_1)}{\boxdot} \mathcal{B}) \to \mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}) : \xi \mapsto (\mathrm{id} \otimes h_{\mathbb{H}_1} \otimes \mathrm{id})V_{12}^*(\mathrm{id} \otimes T)\xi$$

and $\Delta' : L^2(\mathcal{O}(\mathbb{H}_1)) \to C(\mathbb{H}_1) \otimes L^2(\mathcal{O}(\mathbb{H}_1))$ the representation induced by $\Delta_{\mathbb{H}_1}$ one can prove that

$$\begin{aligned}
(\mathrm{id}_{\mathcal{H}} \otimes \Delta' \otimes \mathrm{id}_{L^2(\mathcal{B})})(V_{12}^*(\mathrm{id} \otimes T)\xi) &= V_{13}^* V_{12}^* (\mathrm{id}_{\mathcal{H}} \otimes \Delta' \otimes \mathrm{id}_{L^2(\mathcal{B})})\big((\mathrm{id} \otimes T)\xi\big) \\
&= V_{13}^* V_{12}^* V_{12}((\mathrm{id} \otimes T)\xi)_{134} \\
&= (V_{12}^*(\mathrm{id} \otimes T)\xi)_{134}
\end{aligned}$$

and hence $\phi'(\xi)_{134} = V_{12}^*(\mathrm{id} \otimes T)\xi$ as Δ' is ergodic. (This follows the argument of proposition 3.3.2.) Hence, it follows that $\phi' = \phi^{-1}$. Moreover, $\phi\tilde{D} = \tilde{D}'\phi$. Finally, we have for $z \in \mathcal{A} \underset{\mathcal{O}(\mathbb{G}_1)}{\boxdot} \mathcal{B}$ and $\eta \in \mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B})$,

$$\begin{aligned}
\lambda(z)\phi(\eta) &= V_{12} z_{13} V_{12}^* (\mathrm{id} \otimes T^{-1})(V_{12}\eta_{13}) \\
&= (\mathrm{id} \otimes T^{-1})(V_{12} z_{13} V_{12}^* V_{12} \eta_{13}) \\
&= (\mathrm{id} \otimes T^{-1})(V_{12} z_{13} \eta_{13}) \\
&= \phi(z\eta)
\end{aligned}$$

completing the proof. $\square$

6.3 Deformation of the quantum isometry group

In this section we prove the main theorem of this chapter. In the following subsection, we investigate how the universal objects in the category $\mathcal{Q}_R(\mathcal{A}, \mathcal{H}, D)$ are transformed under our deformation procedure.

6.3.1 Deformation of the universal object in $\mathcal{Q}_R(\mathcal{A}, \mathcal{H}, D)$

First we construct the deformed operator $\tilde{R}$.

Proposition 6.3.1. *Let R be a positive invertible operator such that $(\mathcal{A}, \mathcal{H}, D)$ is a R-twisted spectral triple. Suppose $\mathbb{G}_1$ is a compact quantum group acting algebraically and by orientation-preserving isometries on $(\mathcal{A}, \mathcal{H}, D)$ with a representation U and suppose $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$ is a monoidal equivalence. Denote by $(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$ the deformed spectral triple (theorem 3.3.8). Then there exists a positive invertible operator $\tilde{R}$ such that $(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$ is an $\tilde{R}$-twisted spectral triple on which $\mathbb{G}_2$ acts by $\tilde{R}$-twisted volume- and orientation-preserving isometries. Moreover, applying the same construction to φ^{-1}, we obtain R again.*

Proof. We can decompose $\mathcal{H}$ as

$$\mathcal{H} = \oplus_{x \in \mathrm{Irred}(\mathbb{G}_1)} \mathcal{H}_x \otimes W_x$$

for some Hilbert spaces W_x where the direct sum is taken over all $x \in \mathrm{Irred}(\mathbb{G}_1)$, all with multiplicity one. As D commutes with the representation U, D is of the form $D = \oplus_{x \in \mathrm{Irred}(\mathbb{G}_1)} \mathrm{id}_{\mathcal{H}_x} \otimes D_x$ where the D_x are operators $W_x \to W_x$. As $\mathbb{G}_1$ acts by R-twisted volume-preserving isometries,

$$(\tau_R \otimes \mathrm{id})(\alpha_U(a)) = \tau_R(a) 1_{C(\mathbb{G})}$$

for all $a \in \mathcal{E}_D$, where $\tau_R(a) = \mathrm{Tr}(Ra)$ and where $\mathcal{E}_D$ is the *-subalgebra of $B(\mathcal{H})$ generated by the rank-one operators of the form $\eta\xi^*$, η, ξ eigenvectors of D. Taking eigenvectors v_k^x resp. w_r^y of D_x resp. D_y, then $a = (\xi_j^x \otimes v_k^x)(\xi_t^y \otimes w_r^y)^* \in \mathcal{E}_D$ and hence:

$$\begin{aligned}
&(\tau_R \otimes h_{\mathbb{G}_1})(\alpha_U(a)) \\
&\quad = (\tau_R \otimes h_{\mathbb{G}_1})\big((U_{13}(\xi_j^x \otimes v_k^x \otimes 1_{C(\mathbb{G})}))(U_{13}(\xi_t^y \otimes w_r^y \otimes 1_{C(\mathbb{G})}))^*\big) \\
&\quad = \sum_{n,m,z,i,s} \langle \xi_n^z \otimes t_m^z, R(\xi_i^x \otimes v_k^x)\rangle \langle \xi_s^y \otimes w_r^y, \xi_n^z \otimes t_m^z\rangle h_{\mathbb{G}_1}(u_{ij}^x (u_{st}^y)^*)
\end{aligned}$$

$$= \sum_{i,s} \langle \xi_s^y \otimes w_r^y, R(\xi_i^x \otimes v_k^x) \rangle \delta_{x,y} \delta_{i,s} \frac{(F_x)_{jt}}{\mathrm{Tr}(F_x)}$$

$$= \sum_{i} \langle \xi_i^x \otimes w_r^x, R(\xi_i^x \otimes v_k^x) \rangle \delta_{x,y} \frac{(F_x)_{jt}}{\mathrm{Tr}(F_x)},$$

where F_x is the matrix such that $h_{\mathbb{G}_1}(u_{ij}^x (u_{st}^y)^*) = \frac{\delta_{x,y}\delta_{i,s}(F_x)_{jt}}{\mathrm{Tr}(F_x)}$ (described by Woronowicz in [105]). As also $(\tau_R \otimes h_{\mathbb{G}_1})(\alpha_U(a)) = \tau_R(a)$, we have

$$\sum_{i} \langle \xi_i^x \otimes w_r^x, R(\xi_i^x \otimes v_k^x) \rangle \delta_{x,y} \frac{(F_x)_{jt}}{\mathrm{Tr}(F_x)} = \langle \xi_t^y \otimes w_r^y, R(\xi_j^x \otimes v_k^x) \rangle \tag{6.3.1}$$

hence, we see that $R(\mathcal{H}_x \otimes W_x) \subset \mathcal{H}_x \otimes W_x$, as if $x \neq y$, then $\langle \xi_t^y \otimes w_r^y, R(\xi_j^x \otimes v_k^x) \rangle = 0$. Let's say $R = \oplus_{x \in \mathrm{Irred}(\mathbb{G}_1)} R'_x$. Moreover, $R'_x = R_x^1 \otimes R_x^2$ as $\Big(\sum_i \langle \xi_i^x \otimes w_r^x, R(\xi_i^x \otimes v_k^x) \rangle \Big) \frac{(F_x)_{jt}}{\mathrm{Tr}(F_x)}$ is the product of two matrix coefficients, one depending on r and k, the other on j and t. Hence, it follows from (6.3.1) that

$$\mathrm{Tr}(R_x^1) \frac{(F_x)_{jt}}{\mathrm{Tr}(F_x)} (R_x^2)_{r,k} = (R_x^1)_{t,j} (R_x^2)_{r,k}.$$

(Note that this argument is analogous to the proof of theorem 3.8 of [53] but not entirely the same.) Hence R must be of the form $R = \oplus_{x \in \mathrm{Irred}(\mathbb{G}_1)} (F_x)^{\mathsf{T}} \otimes R_x$ with $R_x : W_x \to W_x$. Note moreover that as R is positive and invertible (and the operators F_x are positive and invertible) that all operators R_x are positive and invertible as well. As $(\mathcal{A}, \mathcal{H}, D)$ is an R-twisted spectral triple, R and D commute and hence each D_x commutes with R_x for all $x \in \mathrm{Irred}(\mathbb{G}_1)$. Now, in this presentation

$$\tilde{\mathcal{H}} = \mathcal{H} \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}) = \bigoplus_{x \in \mathrm{Irred}(\mathbb{G}_1)} (\mathcal{H}_x \otimes W_x) \underset{C(\mathbb{G}_1)}{\boxtimes} L^2(\mathcal{B}) \cong \bigoplus_{x \in \mathrm{Irred}(\mathbb{G}_1)} (\mathcal{H}_{\varphi(x)} \otimes W_x)$$

by proposition 3.3.2 and $\tilde{D} = \oplus_{x \in \mathrm{Irred}(\mathbb{G}_1)} \mathrm{id}_{\mathcal{H}_{\varphi(x)}} \otimes D_x$. Therefore, define $\tilde{R} = \oplus_{x \in \mathrm{Irred}(\mathbb{G}_1)} F_{\varphi(x)}^{\mathsf{T}} \otimes R_x$. As the operators R_x are positive and invertible and so are the matrices $F_{\varphi(x)}$, then $\tilde{R}$ is positive, and invertible as well. Furthermore, as for all $x \in \mathrm{Irred}(\mathbb{G})$, R_x commute with D_x, we have that $\tilde{R}$ commutes with $\tilde{D}$. Moreover, $\mathbb{G}_2$ acts by $\tilde{R}$-twisted volume preserving isometries: for $\tilde{a} = (\xi_j^{\varphi(x)} \otimes v_k^{\varphi(x)})(\xi_t^{\varphi(y)} \otimes$

$w_r^{\varphi(y)})^* \in \mathcal{E}_D$, we have:

$$\begin{aligned}
&(\tau_{\tilde{R}} \otimes \mathrm{id}_{C(\mathbb{G}_2)})(\alpha_{\tilde{U}}(\tilde{a}))\\
&= (\tau_{\tilde{R}} \otimes \mathrm{id}_{C(\mathbb{G}_2)})\big((\tilde{U}_{13}(\xi_j^{\varphi(x)} \otimes v_k^{\varphi(x)}))(\tilde{U}_{13}(\xi_t^{\varphi(y)} \otimes w_r^{\varphi(y)}))^*\big)\\
&= \sum_{n,m,z,i,s} \langle \xi_n^{\varphi(z)} \otimes t_m^{\varphi(z)}, \tilde{R}(\xi_i^{\varphi(x)} \otimes v_k^{\varphi(x)})\rangle\\
&\qquad \langle \xi_s^{\varphi(y)} \otimes w_r^{\varphi(y)}, \xi_n^{\varphi(z)} \otimes t_m^{\varphi(z)}\rangle u_{ij}^{\varphi(x)}(u_{st}^{\varphi(y)})^*\\
&= \sum_{i,s} \langle \xi_s^{\varphi(y)} \otimes w_r^{\varphi(y)}, F_{\varphi(x)}^{\mathsf{T}}(\xi_i^{\varphi(x)}) \otimes R_x(v_k^{\varphi(x)})\rangle u_{ij}^{\varphi(x)}(u_{st}^{\varphi(y)})^*\\
&= \sum_{i,s} (F_{\varphi(x)})_{i,s}(R_x)_{r,k} u_{ij}^{\varphi(x)}(u_{st}^{\varphi(x)})^* \delta_{x,y}\\
&= (F_{\varphi(x)})_{j,t}(R_x)_{r,k}\delta_{x,y}\\
&= \langle \xi_t^{\varphi(y)} \otimes w_r^{\varphi(y)}, \tilde{R}(\xi_j^{\varphi(x)} \otimes v_k^{\varphi(x)})\rangle\\
&= \tau_{\tilde{R}}(\tilde{a})
\end{aligned}$$

where the fourth equality is found by exploiting that

$$(h_{\mathbb{G}_2} \otimes \mathrm{id}_{C(\mathbb{G}_2)})\Delta_{\mathbb{G}_2}(u_{kj}^{\varphi(x)}(u_{kt}^{\varphi(x)})^*) = h_{\mathbb{G}_2}(u_{kj}^{\varphi(x)}(u_{kt}^{\varphi(x)})^*)1_{C(\mathbb{G}_2)}.$$

By linearity, this holds for every $a \in \mathcal{E}_D$.

Finally, it is clear that the inverse construction gives R again. $\square$

With this result, we can state the following theorem.

Theorem 6.3.2. *Let R be a positive invertible operator on a Hilbert space $\mathcal{H}$ and let $(\mathcal{A}, \mathcal{H}, D)$ be an R-twisted compact spectral triple on which $\mathrm{QISO}_R^0(\mathcal{A}, \mathcal{H}, D)$ acts algebraically. Suppose $\varphi : \mathrm{QISO}_R^0(\mathcal{A}, \mathcal{H}, D) \to \mathbb{G}_2$ is a monoidal equivalence with bi-Galois object $\mathcal{B}$. Then $\mathbb{G}_2 \cong \mathrm{QISO}_{\tilde{R}}^0(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$ for $\tilde{R}$ as in proposition 6.3.1.*

Remark 6.3.3. *Note that the condition that $\mathrm{QISO}_R^0(\mathcal{A}, \mathcal{H}, D)$ acts algebraically on $(\mathcal{A}, \mathcal{H}, D)$ is not essential. If $\mathrm{QISO}_R^0(\mathcal{A}, \mathcal{H}, D)$ does not act algebraically on $(\mathcal{A}, \mathcal{H}, D)$, we know from proposition 3.1.4 that there exists a *-algebra $\mathcal{A}_1$*

which is SOT-dense in $\mathcal{A}''$ such that $(\mathcal{A}_1, \mathcal{H}, D)$ is a compact spectral triple on which $\mathrm{QISO}^0_R(\mathcal{A}, \mathcal{H}, D)$ acts algebraically. Moreover, $\mathrm{QISO}^0_R(\mathcal{A}, \mathcal{H}, D) \cong \mathrm{QISO}^0_R(\mathcal{A}_1, \mathcal{H}, D)$ by proposition 3.9 of [53].

Proof of theorem 6.3.2. By proposition 6.1.4, there exists a universal object $\mathrm{QISO}^0_R(\mathcal{A}, \mathcal{H}, D)$ in the category $\mathcal{Q}_R$ of compact quantum groups acting by R-twisted volume- and orientation preserving isometries on $(\mathcal{A}, \mathcal{H}, D)$. For notational convenience, we will denote this quantum group by $\mathrm{QISO}^0_\mathrm{R}$. Now, as $\varphi : \mathrm{QISO}^0_\mathrm{R} \to \mathbb{G}_2$ is a monoidal equivalence, $\mathbb{G}_2$ acts algebraically and by orientation preserving isometries on $(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D}) = (\mathcal{A} \underset{\mathcal{O}(\mathrm{QISO}^0_\mathrm{R})}{\boxdot} \mathcal{B}, \mathcal{H} \underset{C(\mathrm{QISO}^0_\mathrm{R})}{\boxtimes} L^2(\mathcal{B}), \tilde{D})$. Denote by $\tilde{R}$ the operator constructed in proposition 6.3.1, then $\mathbb{G}_2$ acts $\tilde{R}$-twisted volume-preserving and hence, it is a quantum subgroup of $\mathrm{QISO}^0_{\tilde{R}}(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$. Moreover, the monoidal equivalence $\varphi^{-1} : \mathbb{G}_2 \to \mathrm{QISO}^0_R(\mathcal{A}, \mathcal{H}, D)$ induces a unitary fiber functor ψ' on $\mathrm{QISO}^0_{\tilde{R}}(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$ by proposition 6.2.14; we will denote the deformed quantum group by $\mathbb{H}_1$, the monoidal equivalence associated to ψ' (for notational convenience) by $\varphi'^{-1} : \mathrm{QISO}^0_{\tilde{R}}(\mathcal{A}, \mathcal{H}, D) \to \mathbb{H}_1$ and the associated bi-Galois object by $\tilde{\mathcal{B}}'$. Summarizing this in a diagram, we get

$$\begin{tikzcd} \mathbb{H}_1 & \mathrm{QISO}^0_{\tilde{R}}(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D}) \arrow[l, dashed, "\varphi'^{-1}"'] \\ \mathrm{QISO}_R(\mathcal{A}, \mathcal{H}, D) \arrow[u, phantom, "\geq"] \arrow[r, "\varphi"] & \mathbb{G}_2 \arrow[u, phantom, "\geq"] \end{tikzcd}$$

As $\mathbb{G}_2$ is a quantum subgroup of $\mathrm{QISO}^0_{\tilde{R}}(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$, $\mathrm{QISO}^0_R(\mathcal{A}, \mathcal{H}, D)$ is a quantum subgroup of $\mathbb{H}_1$ by proposition 6.2.16 and both act by R-twisted volume- and orientation-preserving isometries on $(\mathcal{A}, \mathcal{H}, D)$ by proposition 6.2.17. Hence by universality

$$\mathrm{QISO}^0_R(\mathcal{A}_1, \mathcal{H}, D) \cong \mathbb{H}_1. \tag{6.3.2}$$

and therefore also

$$\mathbb{G}_2 \cong \mathrm{QISO}^0_{\tilde{R}}(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D}).$$

This completes the proof. □

6.3.2 Deformation of the quantum isometry group

In this subsection we use subsections 6.2.1 and 6.3.1 to strengthen the result of theorem 6.3.2 to quantum isometry groups.

Theorem 6.3.4. *Let $(\mathcal{A}, \mathcal{H}, D)$ be an R-twisted compact spectral triple such that $\mathrm{QISO}^0_R(\mathcal{A}, \mathcal{H}, D)$ acts algebraically on $(\mathcal{A}, \mathcal{H}, D)$. Suppose moreover that we have a monoidal equivalence*

$$\varphi : \mathrm{QISO}^0_R(\mathcal{A}, \mathcal{H}, D) \to \mathbb{G}_2.$$

Then there exists a monoidal equivalence

$$\varphi' : \mathrm{QISO}_R(\mathcal{A}, \mathcal{H}, D) \to \mathrm{QISO}_{\tilde{R}}(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$$

where $(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$ is the spectral triple obtained by deformation with φ by theorem 3.3.8 and $\tilde{R}$ the operator obtained from proposition 6.3.1.

Remark 6.3.5. *One can make again remark 6.3.3 here.*

Proof of theorem 6.3.4. Denote the universal object of $\mathcal{Q}_R$ for notational convenience by $\mathrm{QISO}^0_{\mathrm{R}} = (C(\mathrm{QISO}^0_{\mathrm{R}}), U_0)$. Analogously $\mathrm{QISO}^0_{\tilde{R}} = \mathrm{QISO}^0_{\tilde{R}}(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$. As $C(\mathrm{QISO}_R) = C^*(\{(f \otimes \mathrm{id})\alpha_U(a) \mid a \in \mathcal{A}, f \in \mathcal{A}^*\})$, it is a Woronowicz C^*-subalgebra of $\mathrm{QISO}^0_{\mathrm{R}}$ and hence we can apply the theory of section 6.2.1. We obtain a compact quantum group $\mathbb{H}_2$ and a monoidal equivalence $\varphi' : \mathrm{QISO}_R \to \mathbb{H}_2$ and it suffices to prove $\mathbb{H}_2 = \mathrm{QISO}_{\tilde{R}}(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$. Note now that as $\mathrm{QISO}^0_{\mathrm{R}}$ acts algebraically on $(\mathcal{A}, \mathcal{H}, D)$, we can decompose $\mathcal{A}$ into spectral subspaces $\mathcal{A}_x$ and define the subset I of $\mathrm{Irred}(\mathrm{QISO}^0_{\mathrm{R}})$ by $I = \{x \in \mathrm{Irred}(\mathrm{QISO}^0_{\mathrm{R}}) \mid \mathcal{A}_x \neq 0\}$. Then we have $C(\mathrm{QISO}_R) = C^*(\{u^x_{ij} \mid x \in I\})$ by definition of QISO_R and $I = \mathrm{Irred}(\mathrm{QISO}_R)$. Hence, $C(\mathbb{H}_2) = C^*(\{u^{\varphi(x)}_{ij} \mid x \in I\})$ and by theorem 7.3 of [42], we know that also $I = \{x \in \mathrm{Irred}(\mathrm{QISO}^0_{\mathrm{R}}) \mid \tilde{\mathcal{A}}_{\varphi(x)} \neq 0\}$. Therefore, we can conclude that $\mathbb{H}_2 = \mathrm{QISO}_{\tilde{R}}(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$.

This concludes the proof.

□

6.4 Deformation of the quantum isometry group of the Podleś sphere

In this last section of chapter 6, we use section 6.3 to find the quantum isometry group of the newly constructed spectral triple in theorem 5.2.1. Therefore we investigate first the quantum isometry group of the Podleś sphere.

Definition 6.4.1 ([78]). *Define B to be the unital *-subalgebra of $C(SU_q(2))$ generated (as *-algebra) by the elements $\alpha^2, \gamma^*\gamma, \gamma^2, \alpha\gamma$ and $\gamma^*\alpha$. The closure of*

B is a Woronowicz C^*-algebra of $SU_q(2)$ and the associated compact quantum group is called $SO_q(3)$.

Moreover, we have the following theorem of Bhowmick and Goswami.

Theorem 6.4.2 ([17]). *Let $S^2_{q,c}$ be the Podleś sphere as defined in section 5.2.1. Then*

$$\mathrm{QISO}_R(\mathcal{O}(S^2_{q,c}), \mathcal{H}, D) \cong SO_q(3).$$

Now we will investigate how the monoidal equivalences of $SO_q(3)$ are induced by those of $SU_q(2)$ in order to apply theorem 6.3.4 to find the quantum isometry group of the spectral triples constructed in theorem 5.2.1.
In the classical situation, we know that $SO(3)$ is a quotient group of $SU(2)$, indeed $SO(3) = SU(2)/\{-1, 1\}$. In the quantum versions this is also true: we can prove that $\mathbb{Z}_2$ is a normal quantum subgroup of $SU_q(2)$ and $SU_q(2)/\mathbb{Z}_2$ equals $SO_q(3)$.
Describe $C(\mathbb{Z}_2)$ as the following *-algebra:

$$C(\mathbb{Z}_2) = \langle 1, a | a^* = a, a^2 = 1\rangle$$

with the quantum group structure:

$$\varepsilon(a) = 1, \qquad \Delta(a) = a \otimes a, \qquad S(a) = a.$$

Then the *-morphism

$$\theta : C_u(SU_q(2)) \to C(\mathbb{Z}_2) : \begin{cases} 1 \mapsto 1 \\ \alpha \mapsto a \\ \gamma \mapsto 0 \end{cases}$$

makes $\mathbb{Z}_2$ a quantum subgroup of $SU_q(2)$. Moreover it is normal, which we prove in the following proposition.

Proposition 6.4.3. *$\mathbb{Z}_2$ is a normal quantum subgroup of $SU_q(2)$. The quotient quantum group $SU_q(2)/\mathbb{Z}_2$ equals $SO_q(3)$.*

Proof. We have that $C(SU_q(2)/\mathbb{Z}_2) = \{b \in C_u(SU_q(2)) | (\mathrm{id} \otimes \theta)\Delta(b) = b \otimes 1_{C_u(\mathbb{Z}_2)}\}$. Looking at α and γ, we have $\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma$ and $\Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma$ and hence

$$(\mathrm{id} \otimes \theta)\Delta(\alpha) = \alpha \otimes a, (\mathrm{id} \otimes \theta)\Delta(\gamma) = \gamma \otimes a.$$

Hence it follows directly that for $b \in \{\alpha^2, \gamma^*\gamma, \gamma^2, \alpha\gamma, \gamma^*\alpha\}$ one has $(\mathrm{id} \otimes \theta)\Delta(b) = b \otimes 1_{C(\mathbb{Z}_2)}$ and as θ and Δ are *-morphisms, this holds for every $b \in SO_q(3)$. Moreover it is easy to see that $(\mathrm{id} \otimes \theta)\Delta(b) \neq b \otimes 1_{C_u(\mathbb{Z}_2)}$ if $b \notin C(SO_q(3))$. This proves the theorem. $\square$

We defined $SO_q(3)$ as coming from a Woronowicz-C^*-subalgebra of $SU_q(2)$ and proved it is a compact quantum quotient group of $SU_q(2)$. Using the theorems of subsections 6.2.1 and 6.2.2, we will construct monoidal equivalences on $SO_q(3)$. Therefore fix a monoidal equivalence between $SU_q(2)$ and a suitable $A_o(F')$ with $\dim(F') \geq 3$. As $SO_q(3) = SU_q(2)/\mathbb{Z}_2$, we find a Woronowicz subalgebra $I(F')$ of $A_o(F')$ such that $SO_q(3)$ is monoidally equivalent with $I(F')$. Now Theorem 4.1 in [98], gives us a concrete description of $I(F')$.

Theorem 6.4.4 (Theorem 4.1 in [98]). *Let $F \in \mathrm{GL}(n, \mathbb{C})$ be such that $F\overline{F} = \pm I_n$. Then every Woronowicz subalgebra of $A_o(F)$ is a quantum quotient group. Moreover it has only one normal subgroup of order 2 with quantum quotient group $C^*(r_{2m})$ (where r_{2m} is as in the parametrization of Banica [2])*

Applying this theorem to $F = F_q$, it affirms that $SO_q(3)$ is the only compact quantum quotient group of $SU_q(2)$. Applying it to $F = F'$, we get a concrete description of $I(F')$. By remark 5.1.8, it can be seen that the induced monoidal equivalence is not dimension-preserving and hence not a 2-cocycle deformation (by proposition 4.1.2).

Summarizing, we get

Theorem 6.4.5. *Let $F \in \mathrm{GL}(n, \mathbb{C})$ be such that $F\overline{F} = \pm I_n$ and $\varphi : SU_q(2) \to A_o(F)$ a monoidal equivalence with bi-Galois object $\mathcal{B} = A_o(F_q, F)$. Define $I(F)$ to be the C^*-algebra generated by the $U_{ij}U_{kl}$ where U is the unitary in $M_n(A_o(F))$ satisfying the relation $U = F\overline{U}F^{-1}$ as in definition 5.1.1. Define $P(F_q, F)$ to be the *-algebra generated by the $Y_{ij}Y_{kl}$ where Y is the unitary in $M_{2,n}(\mathbb{C}) \otimes \mathcal{O}(A_o(F_q, F))$ described in theorem 5.1.7. Then there exists a monoidal equivalence $\varphi' : SO_q(3) \to I(F)$ with bi-Galois object $\mathcal{B}' = P(F_q, F)$ which is not dimension-preserving (by remark 5.1.8).*

Now we are ready to characterize the quantum isometry groups of the spectral triples constructed in theorem 5.2.1.

Theorem 6.4.6. *Let $q \in (-1, 1) \setminus \{0\}$ and n a natural number with $3 \leq n \leq |q + 1/q|$. If $q > 0$, suppose n is even. With the matrix F defined as in theorem 5.2.1, $I(F)$ as constructed in theorem 6.4.5 is the quantum isometry group of the spectral triple*

$$\Big(\mathcal{O}(S^2_{q,c}) \underset{\mathcal{O}(SU_q(2))}{\boxdot} \mathcal{O}(A_o(F_q, F)), \quad \mathcal{H} \underset{C(SU_q(2))}{\boxtimes} L^2\big(\mathcal{O}(A_o(F_q, F))\big), \quad \tilde{D}\Big)$$

from theorem 5.2.1.

6.5 Conclusion

In the sixth and last chapter, we focus on quantum isometry groups and the question whether the quantum isometry group of a deformed spectral triple is a deformation of the quantum isometry group of the original spectral triple. In the first section, we recall some notions related to quantum isometry groups. In the second, we develop some tools to, given a monoidal equivalence $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$, construct monoidal equivalences on certain Woronowicz C^*-subalgebras of $\mathbb{G}_1$ and $\mathbb{G}_2$ and on certain supergroups of $\mathbb{G}_1$ and $\mathbb{G}_2$. With those tools at hand, in the third section we prove the announced result: the quantum isometry group of a deformed spectral triple is a deformation of the quantum isometry group of the original spectral triple. Finally, in the last section, we apply the results to the example of the Podleś sphere, elaborated in section 5.2.

With the theorems and results obtained in this chapter, we believe this can be a tool to calculate quantum isometry groups of spectral triples of which it is still unknown. In some sense, the monoidal deformation we developed, could be used to deform 'difficult' spectral triples (in the sense that the calculation of the quantum isometry group is difficult) into 'more easy' spectral triples (in the sense that the quantum isometry can be calculated more easily). The following phased plan could be used:

- Find a quantum group $\mathbb{G}$ acting algebraically and by orientation-preserving isometries on the spectral triple.
- Find a monoidal equivalence φ between $\mathbb{G}$ and another quantum group $\mathbb{G}'$.
- Deform the spectral triple with this monoidal equivalence.
- Find the quantum isometry group $\mathbb{H}'$ of this deformed spectral triple. If not possible, find a new monoidal equivalence and/or new quantum group.
- Use the induction methods of section 6.2 on φ^{-1} to find a unitary fiber functor on $\mathbb{H}'$.
- The unitary fiber functor induces a new quantum group $\mathbb{H}$ and a monoidal equivalence $\varphi' : \mathbb{H} \to \mathbb{H}'$. $\mathbb{H}$ is the quantum isometry group of the original spectral triple.

However, it is still unclear how efficient this phased plan can be: how to find a quantum group with a monoidal equivalence to deform a 'difficult' spectral triple in a more easy one? The presentation of the deformed spectral triple does not seem to make it easy to find the quantum isometry group of that spectral triple. Nevertheless, it could be a good technique in some particular cases.

Conclusion and prospects

In this thesis, we have introduced a new deformation procedure for spectral triples, generalizing the procedure developed by Goswami and Joardar in [53]. This procedure uses the symmetries of a non-commutative geometry to deform it: the deformation data consist of a spectral triple, a compact quantum group acting on it algebraically and by orientation preserving isometries and a unitary fiber functor on the quantum group. Moreover, the deformation of the quantum isometry group of a spectral triple is the quantum isometry group of the deformed spectral triple.

The idea of using the symmetries of a certain geometry to deform it, was already present in the approach of Rieffel (where $\mathbb{R}^d$ has an isometric action on the algebra) and in the approach of Goswami-Joardar (where a cocycle on the dual of a compact quantum group which acts isometrically on the spectral triple, is used). In our procedure as well, we use an intrinsic property of the (quantum) symmetries (in the form of a compact quantum group), i.e. its strict monoidal representation category, as a tool to construct a deformation of the spectral triple as well as the quantum group.

We believe this to be an interesting step in the study of spectral triples and quantum isometry groups. The main advantages of this method are in our opinion the following:

- This procedure makes it possible to construct new examples of spectral triples which were not yet found. Our example of the deformed Podleś sphere is a first example in this direction of a spectral triple that was yet unknown.
- Our procedure makes it possible to find the quantum isometry group of spectral triples of whom, until now, no explicit description of the quantum

isometry group was found. We suggested a phased plan to do so in the conclusion of chapter 6.

To have insight in the true values of these new possibilities, further investigation is needed.

To end the general conclusions, we want to give some ideas for future research.

A first and surely interesting approach is the construction of new examples. As our deformation procedure constructs spectral triples, one can use it to find spectral triples with new properties, or new examples of spectral triples with rare properties.

Closely associated to that is the following question: which properties of spectral triples will be preserved by our deformation procedure and which may change? These properties can be properties of the algebra $\mathcal{A}$ (or its C^*-closure) of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ (e.g. nuclearity, existence of non trivial projections, K-theoretic information,...) or properties of the spectral triple itself (like index theoretic information,...).

A third topic is, as announced before, the calculation of quantum isometry groups. One can investigate how to find an 'easy' deformation (i.e. a spectral triple of which the quantum isometry group can easily be calculated) of a 'difficult' spectral triple (i.e. a spectral triple of which the quantum isometry group is difficult to calculate). The phased plan elaborated in the conclusion of chapter 6 then helps to find the quantum isometry group of the 'difficult' spectral triple.

A fourth possible topic for further research is less direct and requires some new steps. However, it is, in our opinion, not less interesting. The question can be posed whether our technique could be lifted to non-compact spectral triples and non-compact quantum groups. Some puzzle pieces are already there:

- Galois objects on locally compact quantum groups have already been developed by De Commer in [40];
- Non-compact (locally compact) spectral triples are investigated in [30,33,48].

Until now, it is not known how a locally compact quantum group can act isometrically on a locally compact quantum manifold. Nevertheless, our feeling is that this is not an impossible challenge and could be a very interesting topic for research. In our opinion, the existence of a locally compact quantum isometry group is more difficult, as the strategy of Bhowmick-Goswami on compact quantum groups will not be applicable. However this is not needed to lift the deformation method to the locally compact level.

Finally, one could ask questions with respect to the applicability in the realm of the non-commutative geometry approach of the standard model. Alain Connes with several coauthors tried to describe the standard model in the framework of non-commutative geometry, e.g. in [34, 35, 44]. Bhowmick et al. investigated the quantum isometry group of this description as a spectral triple in [14]. Hence, one can ask if our method might be interesting for use in the description of the standard model.

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FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS
SECTION OF ANALYSIS
Celestijnenlaan 200B
B-3001 Leuven